

# Calabi-Yau Manifolds, Hermitian Yang-Mills Instantons and Mirror Symmetry

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## ABSTRACT

We formulate six-dimensional Euclidean gravity as  $SU(4) \cong SO(6)$  Yang-Mills gauge theory. For that purpose we devise a six-dimensional version of the 't Hooft symbols which realizes the isomorphism between  $SO(6)$  Lorentz algebra and  $SU(4)$  Lie algebra. As the  $SO(6)$  Lorentz algebra has two irreducible spinor representations, there are accordingly two kinds of the 't Hooft symbols depending on the chirality of  $SO(6)$  Weyl representation, which leads to a topological classification of Riemannian manifolds according to the Euler characteristic. The Kähler condition can be imposed on the 't Hooft symbols which are projected to  $U(3)$ -valued ones and results in the reduction of the gauge group from  $SU(4)$  to  $U(3)$ . After imposing the Ricci-flat condition, the gauge group in the Yang-Mills gauge theory is further reduced to  $SU(3)$ . Consequently, we find that six-dimensional Calabi-Yau manifolds are equivalent to Hermitian Yang-Mills instantons in  $SU(3)$  Yang-Mills gauge theory. The classification of six-dimensional Riemannian manifolds according to the chirality of  $SO(6)$  Weyl representation leads to an interesting picture about the mirror symmetry of Calabi-Yau manifolds and its generalization.

Keywords: Calabi-Yau manifold, Hermitian Yang-Mills instanton, Mirror symmetry

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# 1 Introduction

String theory predicts [1, 2] that six-dimensional Riemannian manifolds have to play an important role in explaining our four-dimensional Universe. They serve as an internal geometry of string theory with 6 extra dimensions and their shapes and topology determine a detailed structure of the multiplets for elementary particles and gauge fields through the compactification, which leads to a low-energy phenomenology in four dimensions. In particular, a Calabi-Yau manifold, which is a compact, Kähler manifold with vanishing Ricci curvature and so a solution of the Einstein equations without matters, has a prominent role in superstring theory and has been a central focus in both contemporary mathematics and mathematical physics. As the holonomy group of Calabi-Yau manifolds is  $SU(3)$ , a compactification onto the Calabi-Yau manifold in superstring theory preserves  $\mathcal{N} = 1$  supersymmetry in four dimensions. One of the most interesting features in the Calabi-Yau compactification is an equivalence between type II string theories compactified on distinct Calabi-Yau manifolds. String theories compactified on these two manifolds can lead to identical effective field theories. This suggests that Calabi-Yau manifolds exist in mirror pairs  $M$  and  $\widetilde{M}$  where the number of vector multiplets  $h^{1,1}(M)$  on a Calabi-Yau manifold  $M$  is the same as the number of hypermultiplets  $h^{2,1}(\widetilde{M})$  on another Calabi-Yau manifold  $\widetilde{M}$  and vice versa. This duality between two Calabi-Yau manifolds is known as the mirror symmetry [3]. While many beautiful properties of the mirror symmetry have been discovered, it is fair to say that we do not have a deep understanding of the reason for the existence of mirror symmetry.

In order to address the mirror symmetry in a different angle, we will formulate six-dimensional Euclidean gravity as a gauge theory. In general relativity, the Lorentz group appears as the structure group acting on orthonormal frames in the tangent space of a Riemannian manifold  $M$  [4]. On a Riemannian manifold  $M$  of dimension 6, the spin connection  $\omega$  is an  $SO(6)$  gauge field. To be precise, under a local Lorentz transformation which is the orthogonal rotation in  $SO(6)$ , a matrix-valued spin connection  $\omega_{AB} = \omega_{MAB} dx^M$  plays a role of gauge fields in  $SO(6)$  gauge theory. From the  $SO(6)$  gauge theory point of view, the Riemann curvature tensors precisely correspond to the field strengths of the  $SO(6)$  gauge fields  $\omega_{AB}$ . Since the Lie algebra of  $SO(6)$  is isomorphic to that of  $SU(4)$ , six-dimensional Euclidean gravity may be formulated as an  $SU(4) \cong SO(6)$  Yang-Mills gauge theory. In this gauge theory approach, the  $SO(6)$  holonomy group of a six-dimensional Riemannian manifold  $M$  plays a role of gauge group.

Via the gauge theory formulation of six-dimensional Euclidean gravity, we want to find gauge theory objects corresponding to Calabi-Yau manifolds and to understand their mirror symmetry in terms of Yang-Mills gauge theory. To get some insight, it will be useful to address the same problem in four-dimensional situation, which is comparatively simple. For the four-dimensional case, see, for example, [5, 6] and references therein. In four dimensions, the Euclidean gravity can be formulated as an  $SO(4) = SU(2)_L \times SU(2)_R$  gauge theory. The four-dimensional space has mystic features. The group  $SO(4)$  for  $d \geq 3$  is the only non-simple Euclidean Lorentz group and one can define a

self-dual two-form only for  $d = 4$ . The Hodge  $*$ -operator acts on a vector space  $\Lambda^p T^*M$  of  $p$ -forms and defines an automorphism of  $\Lambda^2 T^*M$  with eigenvalues  $\pm 1$ . Therefore, we have the decomposition

$$\Lambda^2 T^*M = \Lambda_3^+ \oplus \Lambda_3^- \quad (1.1)$$

where  $\Lambda_3^\pm \equiv P_\pm \Lambda^2 T^*M$  and  $P_\pm = \frac{1}{2}(1 \pm *)$ . The above Hodge decomposition can harmoniously be incorporated with the Lie group homomorphism  $SO(4) = SU(2)_L \times SU(2)_R$ . In this respect, the 't Hooft symbols  $\eta_{AB}^a$  and  $\bar{\eta}_{AB}^{\dot{a}}$  take a superb mission consolidating the Hodge decomposition (1.1) and the Lie algebra isomorphism  $SO(4) = SU(2)_L \times SU(2)_R$ , which intertwines the group structure carried by the Lie algebra indices  $a = 1, 2, 3 \in SU(2)_L$  and  $\dot{a} = 1, 2, 3 \in SU(2)_R$  with the spacetime structure of two-form indices  $A, B$ .

As Riemann curvature tensors are  $SO(4)$ -valued two-forms, one can define the self-dual structure according to the decomposition (1.1). The eigenspace  $\Lambda_3^+$  or  $\Lambda_3^-$  in Eq. (1.1) is called a gravitational (anti-)instanton in this case. It can be shown that the gravitational instanton is a Ricci-flat Kähler manifold and so a Calabi-Yau 2-fold. Therefore, gravitational instantons are hyper-Kähler manifolds with  $SU(2)$  holonomy. It can be shown [5, 6] that gravitational instantons satisfy exactly the same self-duality equation of  $SU(2)$  Yang-Mills instantons on the Ricci-flat manifold determined by the gravitational instantons themselves and so any gravitational instanton is an  $SU(2)$  Yang-Mills instanton on a Ricci-flat four-manifold from the gauge theory point of view.

The spinor representation of  $SO(4)$  is reducible and there are two irreducible Weyl representations. The two irreducible spinor representations are given by an  $SU(2)_L$  spinor  $\mathbf{2} = S_+$  and an  $SU(2)_R$  spinor  $\bar{\mathbf{2}} = S_-$ . (Because the  $SU(2)$  group has only a real representation,  $\bar{\mathbf{2}}$  does not mean a complex conjugate of  $\mathbf{2}$  but a completely independent spinor.) According to the correspondence [7] between the Clifford algebra  $\mathbb{Cl}(4)$  and the exterior algebra  $\Lambda^*M = \bigoplus_{k=0}^4 \Lambda^k T^*M$  (whose six-dimensional version is shown up in Eq (3.2)), the eigenspaces  $\Lambda_3^+$  and  $\Lambda_3^-$  in Eq. (1.1) can be identified with the tensor products  $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$  and  $\bar{\mathbf{2}} \otimes \bar{\mathbf{2}} = \bar{\mathbf{3}} \oplus \bar{\mathbf{1}}$  with singlets being removed, respectively. Since the 't Hooft symbols  $\eta_{AB}^a$  and  $\bar{\eta}_{AB}^{\dot{a}}$  have a one-to-one correspondence with the spaces  $\Lambda_3^+$  and  $\Lambda_3^-$  in Eq. (1.1), respectively, one can see that  $\eta_{AB}^a \in \mathbf{3}$  and  $\bar{\eta}_{AB}^{\dot{a}} \in \bar{\mathbf{3}}$ . As a result, a gravitational instanton can be represented by the basis  $\eta_{AB}^a \in \mathbf{3}$  and it lives in the positive-chirality space  $S_+ = \mathbf{2}$  while an anti-gravitational instanton can be represented by the basis  $\bar{\eta}_{AB}^{\dot{a}} \in \bar{\mathbf{3}}$  and it lives in the negative-chirality space  $S_- = \bar{\mathbf{2}}$  [5, 6].

It would be remarked that an instanton and an anti-instanton in gravity or gauge theory or whatever should be regarded as independent and pairwise components. Thus it will be interesting to consider the behavior of topological invariants under the exchange  $\mathbf{2}$  (instanton)  $\leftrightarrow \bar{\mathbf{2}}$  (anti-instanton). Interestingly, this exchange might be used to define a four-dimensional version of mirror symmetry [8]. Suppose that  $M$  is a four-dimensional Einstein manifold which is simply related to another Einstein manifold  $\widetilde{M}$  by the *mirror* transformation  $\mathbf{2}$  (instanton)  $\leftrightarrow \bar{\mathbf{2}}$  (anti-instanton). Then one can show [5, 6] that the topological invariants on  $M$ , the Euler characteristic  $\chi(M)$  and the Hirzebruch

signature  $\tau(M)$ , behave under the mirror transformation as follows

$$\chi(M) = \chi(\widetilde{M}) \geq 0, \quad \tau(M) = -\tau(\widetilde{M}). \quad (1.2)$$

It is remarkable that the mirror symmetry (1.2) holds for not only Calabi-Yau 2-folds but also any four-dimensional (compact) Einstein manifolds.

Now let us return to the six-dimensional case. Of course, some acute changes arise. First of all, a self-dual two-form cannot be defined by itself. Another contrast is that the Lorentz group  $SO(6) \cong SU(4)$  is a simple group unlike  $SO(4)$ . In addition, for the topological invariant in six dimensions, the Hirzebruch signature  $\tau(M)$  cannot be defined because it is defined only in  $4k$  dimensions. Thereby the mirror symmetry in six dimensions needs to be defined in a different way, as we already know. Nevertheless, it turns out that there is an intimate similarity between the four- and six-dimensional mirror symmetries. A useful access is first to identify the gauge theory object corresponding to a Calabi-Yau 3-fold like as a Calabi-Yau 2-fold has been identified with an  $SU(2)$  Yang-Mills instanton in four dimensions. Remarkably, there exists such a natural object so-called Hermitian Yang-Mills instanton which is a six-dimensional generalization of the four-dimensional Yang-Mills instanton. And this identification has been well-known to string theorists and mathematicians. We quote a paragraph in [9] (211 page) to vividly summarize this picture.

The point of intersection between the Calabi conjecture and the DUY theorem is the tangent bundle. And here's why: Once you've proved the existence of Calabi-Yau manifolds, you have not only those manifolds but their tangent bundles as well, because every manifold has one. Since the tangent bundle is defined by the Calabi-Yau manifold, it inherits its metric from the parent manifold (in this case, the Calabi-Yau). The metric for the tangent bundle, in other words, must satisfy the Calabi-Yau equations. It turns out, however, that for the tangent bundle, the Hermitian Yang-Mills equations are the same as the Calabi-Yau equations, provided the background metric you've selected is the Calabi-Yau. Consequently, the tangent bundle, by virtue of satisfying the Calabi-Yau equations, automatically satisfies the Hermitian Yang-Mills equations, too.

If a Calabi-Yau manifold  $M$  can be identified with a Hermitian Yang-Mills instanton, a natural question immediately arises. Since Calabi-Yau manifolds exist in mirror pairs, there will be a mirror Calabi-Yau manifold  $\widetilde{M}$  obeying the mirror relation  $h^{1,1}(M) = h^{2,1}(\widetilde{M})$ ,  $h^{2,1}(M) = h^{1,1}(\widetilde{M})$ . This in turn implies that there should be a *mirror* Hermitian Yang-Mills instanton which can be identified with the mirror Calabi-Yau manifold  $\widetilde{M}$ . What is then the relation between the Hermitian Yang-Mills instanton and its mirror Hermitian Yang-Mills instanton?

An essential hint may be obtained by invoking the four-dimensional mirror transformation defined by  $\mathbf{2}(\text{instanton}) \leftrightarrow \overline{\mathbf{2}}(\text{anti-instanton})$ . In this mirror transformation,  $\mathbf{2}(\text{instanton})$  means the positive chirality spinor of  $SO(4)$ , i.e. an  $SU(2)_L$  spinor space, in which a Calabi-Yau 2-fold or an  $SU(2)$  Yang-Mills instanton lives, while  $\overline{\mathbf{2}}(\text{anti-instanton})$  means the negative chirality spinor

of  $SO(4)$ , i.e. an  $SU(2)_R$  spinor space, in which a mirror Calabi-Yau 2-fold or an  $SU(2)$  Yang-Mills anti-instanton lives. In this correspondence, the  $SU(2)$  gauge group of Yang-Mills instantons is identified with the holonomy group of Calabi-Yau 2-folds. With a clever guess, an extension to six dimensions is somewhat obvious. First,  $\mathbf{2}$  will be replaced by  $\mathbf{3}$  because, instead of a Calabi-Yau 2-fold, there is a Calabi-Yau 3-fold whose holonomy group is  $SU(3)$  and the  $SU(2)$  Yang-Mills instanton will be replaced by an  $SU(3)$  Hermitian Yang-Mills instanton. If true, it will be worthwhile to recall that the fundamental representation of  $SU(3)$  is a complex representation and so the complex conjugate  $\bar{\mathbf{3}}$  of a complex representation  $\mathbf{3}$  is a different and inequivalent representation. Therefore, one can embed the mirror Hermitian Yang-Mills instanton into the anti-fundamental representation  $\bar{\mathbf{3}}$ . This structure may be summarized with a schematic form:

$$\begin{array}{ccc}
CY3(M) & \longrightarrow & CY3(\widetilde{M}) \\
\downarrow & & \downarrow \\
HYM(\mathbf{3}) & \longrightarrow & HYM(\bar{\mathbf{3}})
\end{array} \tag{1.3}$$

Here  $CY3(M)$  refers to a Calabi-Yau manifold  $M$  and  $CY3(\widetilde{M})$  its mirror. And  $HYM(\mathbf{3})$  refers to a Hermitian Yang-Mills instanton in the fundamental representation  $\mathbf{3}$  of  $SU(3)$  and  $HYM(\bar{\mathbf{3}})$  its mirror in the anti-fundamental representation  $\bar{\mathbf{3}}$ .

The purpose of this paper is to understand the structure in the diagram (1.3). We will formulate six-dimensional Euclidean gravity as  $SU(4) \cong SO(6)$  Yang-Mills gauge theory. Since the  $SU(3)$  holonomy group of Calabi-Yau manifolds appears as a gauge group in the gauge theory formulation, it is natural to consider the  $SU(3)$  gauge group in the Hermitian Yang-Mills instanton as a subgroup of the original  $SU(4) \cong SO(6)$  gauge group. It is noted that, as the  $SO(6)$  Lorentz algebra is isomorphic to the Lie algebra of  $SU(4)$  and  $SO(6)$  has two irreducible spinor representations, the positive- and negative-chirality spinors of  $SO(6)$  can be identified with the fundamental representation  $\mathbf{4}$  and the anti-fundamental representation  $\bar{\mathbf{4}}$  of  $SU(4)$ , respectively. Thereby, if the structure in the diagram (1.3) is true, it implies that a mirror pair of two Calabi-Yau manifolds in different Weyl spinor representations can be understood as a mirror pair of two Hermitian Yang-Mills instantons in different fundamental representations. We will show that this inference is true.

This paper is organized as follows. In Section 2, we will formulate  $d$ -dimensional Euclidean gravity as  $SO(d)$  Yang-Mills gauge theory. The explicit relation between gravity and gauge theory variables will be established.

We will apply in Section 3 the gauge theory formulation of Euclidean gravity to six-dimensional Riemannian manifolds. For that purpose we will devise a six-dimensional version of the 't Hooft symbols which realizes the isomorphism between  $SO(6)$  Lorentz algebra and  $SU(4)$  Lie algebra. As the  $SO(6)$  Lorentz algebra has two irreducible spinor representations, there are accordingly two kinds of the 't Hooft symbols depending on the chirality of  $SO(6)$  Weyl representation. This leads

to a topological classification of six-dimensional Riemannian manifolds according to the Euler characteristic whose sign is correlated with the six-dimensional chirality. The Kähler condition can be imposed on the 't Hooft symbols which are projected to  $U(3)$ -valued ones and results in the reduction of the gauge group from  $SU(4)$  to  $U(3)$ . After imposing the Ricci-flat condition, the gauge group in the Yang-Mills gauge theory is further reduced to  $SU(3)$ . Consequently, we find that six-dimensional Calabi-Yau manifolds are equivalent to Hermitian Yang-Mills instantons in  $SU(3)$  Yang-Mills gauge theory.

In Section 4, we will explore the geometrical properties of Calabi-Yau manifolds in the positive- and negative-chirality representations of  $SO(6)$ . We construct cohomology classes in each chiral representation and find a mirror relation in their representation acting on spinor states  $S_{\pm}$  of definite chirality. We show that the Euler characteristic of Calabi-Yau manifolds in different chiral representations of  $SO(6)$  has an opposite sign consistent with the mirror symmetry.

In Section 5, we will again derive the relation between Calabi-Yau manifolds and Hermitian Yang-Mills instantons and then discuss the mirror symmetry between Calabi-Yau manifolds from a completely gauge theory setup. We find that a mirror Calabi-Yau manifold corresponds to a Hermitian Yang-Mills instanton in a different complex representation  $\mathbf{3}$  or  $\bar{\mathbf{3}}$  of  $SU(3) \subset SU(4)$ . As a result, the integral of the third Chern class  $c_3(E)$  for the gauge bundle  $E$  which is equal to the Euler characteristic in the case of tangent bundle, has an inevitable sign flip between the fundamental representation  $\mathbf{3}$  and the anti-fundamental representation  $\bar{\mathbf{3}}$ .

Finally we will discuss in Section 6 the results obtained in this paper with some remarks about some generalization of mirror symmetry.

In Appendix A, we fix the basis for the chiral representation of  $SO(6)$  and the fundamental representation of  $SU(4)$  and list their structure constants. In Appendix B, we present an explicit matrix representation and their algebras of the six-dimensional 't Hooft symbols in each chiral basis.

## 2 Gravity As A Gauge Theory

On a Riemannian manifold  $M$  of dimension  $d$ , the spin connection  $\omega$  is an  $SO(d)$ -valued one-form and can be identified, in general, with an  $SO(d)$  gauge field. In order to make an explicit identification between the spin connections and the corresponding gauge fields, let us first introduce the  $d$ -dimensional Clifford algebra

$$\{\Gamma^A, \Gamma^B\} = 2\delta^{AB}, \quad (2.1)$$

where  $\Gamma^A$  ( $A = 1, \dots, d$ ) are Dirac matrices. Then the  $SO(d)$  Lorentz generators are given by

$$J^{AB} = \frac{1}{4}[\Gamma^A, \Gamma^B] \quad (2.2)$$

which satisfy the following Lorentz algebra

$$[J^{AB}, J^{CD}] = -(\delta^{AC} J^{BD} - \delta^{AD} J^{BC} - \delta^{BC} J^{AD} + \delta^{BD} J^{AC}). \quad (2.3)$$



The  $SO(d)$ -valued spin connection is defined by  $\omega = \frac{1}{2}\omega_{AB}J^{AB}$  where  $\omega_{AB} = \omega_{MAB}dx^M$  are connection one-forms on  $M$ , which transforms in the standard way as an  $SO(d)$  gauge field under local Lorentz transformations

$$\omega \rightarrow \omega' = \Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1} \quad (2.4)$$

where  $\Lambda = e^{\frac{1}{2}\lambda_{AB}(x)J^{AB}} \in SO(d)$ .

Now we introduce an  $SO(d)$ -valued gauge field defined by  $A = A^a T^a$  where  $A^a = A_M^a dx^M$  ( $a = 1, \dots, \frac{d(d-1)}{2}$ ) are connection one-forms on  $M$  and  $T^a$  are Lie algebra generators of  $SO(d)$  satisfying

$$[T^a, T^b] = -f^{abc}T^c. \quad (2.5)$$

The identification we want to make is then given by

$$\omega = \frac{1}{2}\omega_{AB}J^{AB} \equiv A = A^a T^a. \quad (2.6)$$

One may adopt the identification (2.6) by applying a group homomorphism of  $SO(d)$  such that  $SO(3) = SU(2)$ ,  $SO(4) = SU(2)_L \times SU(2)_R$ ,  $SO(5) = Sp(2)$ , and  $SO(6) = SU(4)$ .<sup>1</sup> Then the Lorentz transformation (2.4) can be translated into a usual gauge transformation

$$A \rightarrow A' = \Lambda A \Lambda^{-1} + \Lambda d\Lambda^{-1} \quad (2.7)$$

where  $\Lambda = e^{\lambda^a(x)T^a} \in SO(d)$ .

The  $SO(d)$ -valued Riemann curvature tensor is defined by

$$\begin{aligned} R &= d\omega + \omega \wedge \omega \\ &= \frac{1}{2}R_{AB}J^{AB} = \frac{1}{2}(d\omega_{AB} + \omega_{AC} \wedge \omega_{CB})J^{AB} \\ &= \frac{1}{4}(R_{MNAB}J^{AB})dx^M \wedge dx^N \\ &= \frac{1}{4}\left[\left(\partial_M \omega_{NAB} - \partial_N \omega_{MAB} + \omega_{MAC}\omega_{NCB} - \omega_{NAC}\omega_{MCB}\right)J^{AB}\right]dx^M \wedge dx^N \end{aligned} \quad (2.8)$$

or, in terms of gauge theory variables, it is given by

$$\begin{aligned} F &= dA + A \wedge A \\ &= F^a T^a = \left(dA^a - \frac{1}{2}f_{bc}^a A^b \wedge A^c\right)T^a \\ &= \frac{1}{2}(F_{MN}^a T^a)dx^M \wedge dx^N \\ &= \frac{1}{2}\left[\left(\partial_M A_N^a - \partial_N A_M^a - f^{abc}A_M^b A_N^c\right)T^a\right]dx^M \wedge dx^N. \end{aligned} \quad (2.9)$$

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<sup>1</sup>To be precise, the spin connection (2.6) is a connection on a spinor bundle induced from the  $SO(d)$ -bundle and the structure group of its fiber is lifted to  $Spin(d)$  for  $d \geq 3$ , a double cover of  $SO(d)$ , according to the short exact sequence of Lie groups:  $1 \rightarrow \mathbb{Z}_2 \rightarrow Spin(d) \rightarrow SO(d) \rightarrow 1$ . Hence the global isomorphism should refer to  $Spin(d)$ . Nevertheless we will not care about the  $\mathbb{Z}_2$ -factor because we are mostly interested in local descriptions (in the level of Lie algebras).

Let us introduce at each spacetime point in  $M$  a local frame of reference in the form of  $d$  linearly independent vectors (vielbeins)  $E_A = E_A^M \partial_M \in \Gamma(TM)$  which are chosen to be orthonormal, i.e.,  $E_A \cdot E_B = \delta_{AB}$  [4]. The frame basis  $\{E_A\}$  defines a dual basis  $E^A = E_M^A dx^M \in \Gamma(T^*M)$  by a natural pairing

$$\langle E^A, E_B \rangle = \delta_B^A. \quad (2.10)$$

The above pairing leads to the relation  $E_M^A E_B^M = \delta_B^A$ . In terms of the non-coordinate (anholonomic) basis in  $\Gamma(TM)$  or  $\Gamma(T^*M)$ , a Riemannian metric can be written as

$$\begin{aligned} ds^2 &= \delta_{AB} E^A \otimes E^B = \delta_{AB} E_M^A E_N^B dx^M \otimes dx^N \\ &\equiv g_{MN}(x) dx^M \otimes dx^N \end{aligned} \quad (2.11)$$

or

$$\begin{aligned} \left( \frac{\partial}{\partial s} \right)^2 &= \delta^{AB} E_A \otimes E_B = \delta^{AB} E_A^M E_B^N \partial_M \otimes \partial_N \\ &\equiv g^{MN}(x) \partial_M \otimes \partial_N. \end{aligned} \quad (2.12)$$

Using the form language where  $d = dx^M \partial_M = E^A E_A$  and  $A = A_M dx^M = A_A E^A$ , the field strength (2.9) of  $SO(d)$  gauge fields in the non-coordinate basis takes the form

$$\begin{aligned} F &= dA + A \wedge A = \frac{1}{2} F_{AB} E^A \wedge E^B \\ &= \frac{1}{2} \left( E_A A_B - E_B A_A + [A_A, A_B] + f_{AB}^C A_C \right) E^A \wedge E^B \end{aligned} \quad (2.13)$$

where we used the structure equation

$$dE^A = \frac{1}{2} f_{BC}^A E^B \wedge E^C. \quad (2.14)$$

The frame basis  $E_A = E_A^M \partial_M \in \Gamma(TM)$  satisfies the Lie algebra under the Lie bracket

$$[E_A, E_B] = -f_{AB}^C E_C \quad (2.15)$$

where

$$f_{ABC} = E_A^M E_B^N (\partial_M E_{NC} - \partial_N E_{MC}) \quad (2.16)$$

are the structure functions in (2.14).

One can also translate the covariant derivative defined by

$$\nabla_M R_{NP} = \partial_M R_{NP} - \Gamma_{MN}^Q R_{QP} - \Gamma_{MP}^Q R_{NQ} + [\omega_M, R_{NP}] \quad (2.17)$$

into the covariant derivative in gauge theory given by

$$D_M F_{NP} = \partial_M F_{NP} - \Gamma_{MN}^Q F_{QP} - \Gamma_{MP}^Q F_{NQ} + [A_M, F_{NP}], \quad (2.18)$$



where  $\Gamma_{MN}^P = \omega_M^A E_A^P E_N^B + E_A^P \partial_M E_N^A$  is the Levi-Civita connection. It is then easy to show [5] that the second Bianchi identity for curvature tensors is transformed into the Bianchi identity for  $SO(d)$  gauge fields:

$$\nabla_{[M} R_{NP]} = 0 \quad \Leftrightarrow \quad D_{[M} F_{NP]} = 0, \quad (2.19)$$

where the bracket  $[MNP] \equiv \frac{1}{3}(MNP + NPM + PMN)$  denotes the cyclic permutation of indices.

### 3 Spinor Representation of Six-dimensional Riemannian Manifolds

From now on, we will apply the gauge theory formulation in the previous section to six-dimensional Riemannian manifolds. For this purpose, the  $SO(6) \cong SU(4)/\mathbb{Z}_2$  Lorentz group for Euclidean gravity will be identified with the  $SU(4)$  gauge group in Yang-Mills gauge theory. Via the gauge theory formulation of six-dimensional Euclidean gravity, we want to find gauge theory objects corresponding to Calabi-Yau manifolds and to understand their mirror symmetry in terms of Yang-Mills gauge theory. Because our formulation of six-dimensional gravity in terms of Yang-Mills gauge theory is based on the identification (2.6) (i.e., the spin connection instead of the Levi-Civita connection), it is essential to consider a spinor representation of  $SO(6)$  to realize the relationship.

Let us start with the Clifford algebra  $\mathbb{Cl}(6)$  whose generators are given by

$$\mathbb{Cl}(6) = \{\mathbf{I}_8, \Gamma^A, \Gamma^{AB}, \Gamma_{\pm}^{ABC}, \Gamma^7 \Gamma^{AB}, \Gamma^7 \Gamma^A, \Gamma^7\} \quad (3.1)$$

where  $\Gamma^A$  ( $A = 1, \dots, 6$ ) are six-dimensional Dirac matrices satisfying the algebra (2.1),  $\Gamma^{A_1 A_2 \dots A_k} = \frac{1}{k!} \Gamma^{[A_1} \Gamma^{A_2} \dots \Gamma^{A_k]}$  with the complete antisymmetrization of indices,  $\Gamma^7 = i\Gamma^1 \dots \Gamma^6$  is the chiral matrix given by (A.3) and  $\Gamma_{\pm}^{ABC} = \frac{1}{2}(\mathbf{I}_8 \pm \Gamma^7) \Gamma^{ABC}$ . It will be useful to notice [2, 7] that the Clifford algebra (3.1) can be identified with the exterior algebra of a cotangent bundle  $T^*M \rightarrow M$

$$\mathbb{Cl}(6) \cong \Lambda^* M = \bigoplus_{k=0}^6 \Lambda^k T^* M \quad (3.2)$$

where the chirality  $\Gamma^7$  corresponds to the Hodge operator  $*$ :  $\Lambda^k T^* M \rightarrow \Lambda^{6-k} T^* M$ .

The spinor representation of  $SO(6)$  can be constructed by 3 fermion creation operators  $a_i^*$  ( $i = 1, 2, 3$ ) and the corresponding annihilation operators  $a^j$  ( $j = 1, 2, 3$ ) (see Appendix 5.A in [1]). This fermionic system can be represented in a Hilbert space  $V$  of dimension 8 whose states are obtained by acting on a Fock vacuum  $|\Omega\rangle$ , annihilated by all the annihilation operators, by the product of  $k$  creation operators  $a_{i_1}^* \dots a_{i_k}^*$ , i.e.

$$V = \bigoplus_{k=0}^3 |\Omega_{i_1 \dots i_k}\rangle = \bigoplus_{k=0}^3 a_{i_1}^* \dots a_{i_k}^* |\Omega\rangle. \quad (3.3)$$

The spinor representation in the Hilbert space  $V$  is reducible, i.e.  $V = S_+ \oplus S_-$ , and there are two irreducible spinor representations  $S_\pm$  each of dimension 4, namely the spinors of positive and negative chirality. If the Fock vacuum  $|\Omega\rangle$  has positive chirality, the positive chirality spinors of  $SO(6)$  are states given by

$$S_+ = \bigoplus_{k \text{ even}} |\Omega_{i_1 \dots i_k}\rangle = |\Omega\rangle + |\Omega_{ij}\rangle = 4 \quad (3.4)$$

while the negative chirality spinors of  $SO(6)$  are those obtained by

$$S_- = \bigoplus_{k \text{ odd}} |\Omega_{i_1 \dots i_k}\rangle = |\Omega_i\rangle + |\Omega_{ijk}\rangle = \bar{4}. \quad (3.5)$$

As the  $SO(6)$  Lorentz algebra is isomorphic to the Lie algebra of  $SU(4)$  and  $SO(6)$  has two irreducible spinor representations, the positive- and negative-chirality spinors of  $SO(6)$  are the fundamental representations  $4$  and the anti-fundamental representation  $\bar{4}$  of  $SU(4)$ , respectively [1]. In other words, since  $SO(6)$  has two inequivalent spinor representations and the  $4$  and  $\bar{4}$  of  $SU(4)$  are inequivalent, the Weyl spinor representations on  $S_\pm$  can be identified with the (anti-)fundamental  $SU(4)$  representations on  $\mathbb{C}^4$ .

One can form a direct product of the fundamental representations  $4$  and  $\bar{4}$  of  $SU(4)$  in order to classify the Clifford generators in Eq. (3.1):

$$4 \otimes \bar{4} = 1 \oplus 15 = \{\Gamma_+, \Gamma_+^{AB}\}, \quad (3.6)$$

$$\bar{4} \otimes 4 = 1 \oplus 15 = \{\Gamma_-, \Gamma_-^{AB}\}, \quad (3.7)$$

$$4 \otimes 4 = 6 \oplus 10 = \{\Gamma_+^A, \Gamma_+^{ABC}\}, \quad (3.8)$$

$$\bar{4} \otimes \bar{4} = 6 \oplus 10 = \{\Gamma_-^A, \Gamma_-^{ABC}\}, \quad (3.9)$$

where  $\Gamma_\pm \equiv \frac{1}{2}(\mathbf{I}_8 \pm \Gamma^7)$ ,  $\Gamma_\pm^A \equiv \Gamma_\pm \Gamma^A$  and  $\Gamma_\pm^{AB} \equiv \Gamma_\pm \Gamma^{AB}$ . Note that  $15$  in Eqs. (3.6) and (3.7) is the adjoint representation of  $SU(4)$  and  $6$  in Eqs. (3.8) and (3.9) is the antisymmetric second-rank tensor of  $SU(4)$  while  $10$  is the symmetric second-rank tensor of  $SU(4)$ . See Appendix A for the chiral representation of  $SO(6)$  and the fundamental representation of  $SU(4)$ .

According to the identification (2.6), we have the following relation

$$R_{AB} = \frac{1}{2} R_{ABCD} J^{CD} = F_{AB}^a (T_1^a \oplus T_2^a) = F_{AB}. \quad (3.10)$$

On the right hand side, the doubling of  $SU(4)$  algebra in four-dimensional representations  $R_1$  and  $R_2$  was considered because the  $SO(6)$  spinor representation on the left hand side is eight-dimensional. We will regard the Riemann tensor  $R_{AB}$  as a linear map acting on the Hilbert space  $V$  in Eq. (3.3). As  $R_{AB}$  contains two gamma matrices, it does not change the chirality of the vector space  $V$ . Therefore, we can represent it in a subspace of definite chirality as either  $R_{AB} : S_+ \rightarrow S_+$  or  $R_{AB} : S_- \rightarrow S_-$ . The former case  $R_{AB} : S_+ \rightarrow S_+$  will take values in  $4 \otimes \bar{4}$  in (3.6) with a singlet being removed

while the latter case  $R_{AB} : S_- \rightarrow S_-$  will take values in  $\overline{4} \otimes 4$  in (3.7) with no singlet. Therefore, there exist two independent identifications defined by

$$\mathbb{A} : \frac{1}{2} R_{ABCD} J_+^{CD} \equiv F_{AB}^{(+a)} (T^a \oplus \mathbf{0}), \quad (3.11)$$

$$\mathbb{B} : \frac{1}{2} R_{ABCD} J_-^{CD} \equiv F_{AB}^{(-a)} (\mathbf{0} \oplus T^a), \quad (3.12)$$

where we distinguish the two classes  $\mathbb{A}$  and  $\mathbb{B}$  depending on the six-dimensional chirality. See Appendix A for explicit chiral representations  $J_{\pm}^{AB}$  of  $SO(6)$ . Because the classes  $\mathbb{A}$  and  $\mathbb{B}$  are now represented by  $4 \times 4$  matrices on both sides, we can take a trace operation for the matrices which leads to the following relations<sup>2</sup>

$$\mathbb{A} : R_{ABCD} = -F_{AB}^{(+a)} \text{Tr} (T^a J_+^{CD}) \equiv F_{AB}^{(+a)} \eta_{CD}^a, \quad (3.13)$$

$$\mathbb{B} : R_{ABCD} = -F_{AB}^{(-a)} \text{Tr} (T^a J_-^{CD}) \equiv F_{AB}^{(-a)} \overline{\eta}_{CD}^a. \quad (3.14)$$

Here we have introduced the six-dimensional analogue of the 't Hooft symbols defined by

$$\eta_{AB}^a = -\text{Tr} (T^a J_+^{AB}), \quad \overline{\eta}_{AB}^a = -\text{Tr} (T^a J_-^{AB}). \quad (3.15)$$

An explicit matrix representation of the six-dimensional 't Hooft symbols and their algebra are presented in Appendix B.

Note that  $F^{(\pm)a} = \frac{1}{2} F_{AB}^{(\pm)a} E^A \wedge E^B$  in Eqs. (3.13) and (3.14) are field strengths of  $SU(4)$  gauge fields defined by Eq. (2.13). Using Eq. (B.8), one can express them as  $F_{AB}^{(+a)} = R_{ABCD} \eta_{CD}^a = \eta_{CD}^a R_{CDAB}$  and  $F_{AB}^{(-a)} = R_{ABCD} \overline{\eta}_{CD}^a = \overline{\eta}_{CD}^a R_{CDAB}$ . One can apply again the same expansion as Eqs. (3.13) and (3.14) to the index pair  $[AB]$  of the Riemann tensor  $R_{CDAB}$ . That is, one can expand the  $SU(4)$  field strengths in the classes  $\mathbb{A}$  and  $\mathbb{B}$  in terms of the basis in Eq. (3.15)

$$\mathbb{A} : F_{AB}^{(+a)} = f_{(++)}^{ab} \eta_{AB}^b, \quad (3.16)$$

$$\mathbb{B} : F_{AB}^{(-a)} = f_{(--)^{ab}} \overline{\eta}_{AB}^b. \quad (3.17)$$

As was pointed out in Eq. (3.2), the Clifford algebra (3.1) can be identified with the exterior algebra  $\Lambda^* M$  and so the 't Hooft symbols in Eq. (3.15) have a one to one correspondence with the basis of two-forms in  $\Lambda^2 T^* M$  for a given orientation. Consequently, we expand the six-dimensional Riemann

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<sup>2</sup>Since the distinction of the class  $\mathbb{A}$  and the class  $\mathbb{B}$  is meaningful only in the eight-dimensional spinor space (3.3), the curvature tensor  $R_{ABCD}$  in the classes  $\mathbb{A}$  and  $\mathbb{B}$  should be understood as the ones defined by Eq. (3.10) obeying the chirality condition  $\Gamma_+ R_{AB} = R_{AB}$  and  $\Gamma_- R_{AB} = R_{AB}$ , respectively, as is obvious from the definition (3.11) and (3.12). Actually, the Riemann curvature tensor  $R_{ABCD}$  in a usual  $SO(6)$ -frame bundle with Levi-Civita connections can be equally expanded in either the basis  $\mathbb{A}$  or  $\mathbb{B}$  because, in this case, the curvature tensor is immune from the chirality. In our case, we will always assume a spinor bundle lifted from the  $SO(6)$ -frame bundle where the structure group of its fiber is  $Spin(6)$ . See the footnote 1.

curvature tensors according to their six-dimensional chirality class into two different ways:

$$\mathbb{A} : \quad R_{ABCD} = f_{(++)}^{ab} \eta_{AB}^a \eta_{CD}^b, \quad (3.18)$$

$$\mathbb{B} : \quad R_{ABCD} = f_{(--)}^{ab} \bar{\eta}_{AB}^a \bar{\eta}_{CD}^b. \quad (3.19)$$

Of course, the index pairs  $[AB]$  and  $[CD]$  in the curvature tensor  $R_{ABCD}$  should have the same chirality structure because the class  $\mathbb{A}$  lives in the vector space of positive-chirality spinors defined by (3.4) while the class  $\mathbb{B}$  lives in the vector space of negative-chirality spinors defined by (3.5).

Note that the Riemann curvature tensor in 6 dimensions has  $225 = 15 \times 15$  components in total which is the number of expansion coefficients in each class. Because the torsion free condition has been assumed for the curvature tensor, the first Bianchi identity  $R_{A[BCD]} = 0$  should be imposed which leads to 120 constraints. After all, the curvature tensor has  $105 = 225 - 120$  independent components which must be equal to the number of remaining expansion coefficients in the classes  $\mathbb{A}$  and  $\mathbb{B}$  after solving the 120 constraints

$$\varepsilon^{ACDEFG} R_{BCDE} = 0. \quad (3.20)$$

It is worthwhile to notice that the curvature tensor automatically satisfies the symmetry property  $R_{ABCD} = R_{CDAB}$  after dictating the first Bianchi identity (3.20). Therefore, one can split the 120 constraints in Eq. (3.20) into  $105 = \frac{15 \times 14}{2}$  conditions imposing the symmetry  $R_{ABCD} = R_{CDAB}$  and extra 15 conditions. This can be clarified by considering the tensor product of  $SU(4)$  [10]

$$15 \otimes 15 = (1 + 15 + 20 + 84)_S \oplus (15 + 45 + \overline{45})_{AS} \quad (3.21)$$

where the first part with 120 components is symmetric and the second part with 105 components is antisymmetric. It is obvious from our construction that  $f_{(\pm\pm)}^{ab} \in 15 \otimes 15$ . The 84 components in the symmetric part is the number of Weyl tensors in six dimensions and the  $21 = 20 + 1$  components are coming from Ricci tensors. The remaining 15 components will be further removed by the first Bianchi identity (3.20) after expelling the antisymmetric components in Eq. (3.21).

One can easily solve the symmetry property  $R_{ABCD} = R_{CDAB}$  with the coefficients obeying

$$f_{(++)}^{ab} = f_{(++)}^{ba}, \quad f_{(--)}^{ab} = f_{(--)}^{ba}, \quad (3.22)$$

which results in 120 components and belongs to the symmetric part in Eq. (3.21). Now the remaining 15 conditions can be imposed by the equations

$$\varepsilon^{ABCDEF} R_{ABCD} = 0. \quad (3.23)$$

It is obvious that Eq. (3.23) gives rise to nontrivial relations only for the coefficients satisfying Eq. (3.22). Finally Eq. (3.23) can equivalently be written via Eqs. (B.9) and (B.10) as the 15 constraints

$$d^{abc} f_{(++)}^{bc} = d^{abc} f_{(--)}^{bc} = 0. \quad (3.24)$$

In the end, there are 105 remaining components for  $f_{(\pm\pm)}^{ab}$  which precisely match with the independent components of Riemann curvature tensor in the classes  $\mathbb{A}$  and  $\mathbb{B}$ .

Let us introduce the following projection operator acting on  $6 \times 6$  antisymmetric matrices defined by

$$P_{\pm}^{ABCD} \equiv \frac{1}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) \pm \frac{1}{8}\varepsilon^{ABCDEF}\bar{I}_{EF} = P_{\pm}^{CDAB} \quad (3.25)$$

where  $\bar{I} \equiv i\sigma^2 \otimes \mathbf{I}_3$ . Because any  $6 \times 6$  antisymmetric matrix of rank 4 spans a four-dimensional subspace  $\mathbb{R}^4 \subset \mathbb{R}^6$ , the operator (3.25) in this case can be written as

$$P_{\pm}^{ABCD} \equiv \frac{1}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) \pm \frac{1}{4}\varepsilon^{ABCD}, \quad (A, B, C, D) \in \mathbb{R}^4, \quad (3.26)$$

and so it is a projection operator for such a rank 4 matrix, i.e.,

$$P_{\pm}^{ABEF}P_{\pm}^{EFCD} = P_{\pm}^{ABCD}, \quad P_{\pm}^{ABEF}P_{\mp}^{EFCD} = 0. \quad (3.27)$$

Note that some combination of antisymmetric rank 4 matrices can give rise to a  $6 \times 6$  antisymmetric matrix of rank 6 which can be transformed into the canonical form  $\bar{I}_{AB}$  by an  $SO(6)$  rotation. In this case, the operator (3.25) is not a projection operator for the rank 6 matrix  $\bar{I}_{AB}$  but it acts as

$$P_{\pm}^{ABCD}\bar{I}_{CD} = \left(\frac{1}{2} \pm 1\right)\bar{I}_{AB}. \quad (3.28)$$

In general, one can deduce by a straightforward calculation the following properties

$$P_{\pm}^{ABEF}P_{\pm}^{EFCD} = P_{\pm}^{ABCD} + \frac{1}{8}\bar{I}_{AB}\bar{I}_{CD}, \quad P_{\pm}^{ABEF}P_{\mp}^{EFCD} = -\frac{1}{8}\bar{I}_{AB}\bar{I}_{CD}. \quad (3.29)$$

Therefore, one can decompose the 't Hooft symbols in Eq. (3.15) into eigenspaces of the operator (3.25):

$$\begin{aligned} l_{AB}^{(+)\hat{a}} \equiv & \left\{ \eta_{AB}^{13} = \frac{i}{2}\sigma^2 \otimes \lambda_1, \quad \eta_{AB}^{14} = \frac{i}{2}\mathbf{I}_2 \otimes \lambda_2, \quad \frac{1}{\sqrt{3}}(\eta_{AB}^8 - \sqrt{2}\eta_{AB}^{15}) = -\frac{i}{2}\sigma^2 \otimes \lambda_3, \right. \\ & \eta_{AB}^6 = \frac{i}{2}\sigma^2 \otimes \lambda_4, \quad \eta_{AB}^7 = -\frac{i}{2}\mathbf{I}_2 \otimes \lambda_5, \quad \eta_{AB}^{11} = \frac{i}{2}\sigma^2 \otimes \lambda_6, \\ & \left. \eta_{AB}^{12} = -\frac{i}{2}\mathbf{I}_2 \otimes \lambda_7, \quad \frac{2}{\sqrt{3}}\left(-\frac{1}{2}\eta_{AB}^3 + \frac{1}{\sqrt{3}}\eta_{AB}^8 + \frac{1}{\sqrt{6}}\eta_{AB}^{15}\right) = -\frac{i}{2}\sigma^2 \otimes \lambda_8 \right\} \quad (3.30) \end{aligned}$$

$$\begin{aligned} m_{AB}^{(+)\hat{a}} \equiv & \left\{ \eta_{AB}^1 = \frac{i}{2}\sigma^1 \otimes \lambda_2, \quad \eta_{AB}^2 = -\frac{i}{2}\sigma^3 \otimes \lambda_2, \quad \eta_{AB}^9 = -\frac{i}{2}\sigma^1 \otimes \lambda_5, \right. \\ & \left. \eta_{AB}^{10} = \frac{i}{2}\sigma^3 \otimes \lambda_5, \quad \eta_{AB}^4 = \frac{i}{2}\sigma^1 \otimes \lambda_7, \quad \eta_{AB}^5 = -\frac{i}{2}\sigma^3 \otimes \lambda_7 \right\}, \quad (3.31) \end{aligned}$$

$$n_{AB}^{(+)\hat{0}} \equiv \left\{ \eta_{AB}^3 + \frac{1}{\sqrt{3}}\eta_{AB}^8 + \frac{1}{\sqrt{6}}\eta_{AB}^{15} = \frac{1}{2}\bar{I}_{AB} = \frac{i}{2}\sigma^2 \otimes \mathbf{I}_3 \right\}, \quad (3.32)$$

where  $\hat{a}, \hat{b} = 1, \dots, 8$  and  $\dot{a}, \dot{b} = 1, \dots, 6$  are  $SU(4)$  indices in the entries of  $l_{AB}^{(+)\hat{a}}$  and  $m_{AB}^{(+)\dot{a}}$ , respectively. They obey the following relations

$$\begin{aligned} P_{-}^{ABCD} l_{CD}^{(+)\hat{a}} &= l_{AB}^{(+)\hat{a}}, & P_{+}^{ABCD} l_{CD}^{(+)\hat{a}} &= 0, \\ P_{-}^{ABCD} m_{CD}^{(+)\dot{a}} &= 0, & P_{+}^{ABCD} m_{CD}^{(+)\dot{a}} &= m_{AB}^{(+)\dot{a}}, \\ P_{-}^{ABCD} n_{CD}^{(+)\dot{0}} &= -\frac{1}{2} n_{AB}^{(+)\dot{0}}, & P_{+}^{ABCD} n_{CD}^{(+)\dot{0}} &= \frac{3}{2} n_{AB}^{(+)\dot{0}}, \end{aligned} \quad (3.33)$$

which can be summarized as

$$P_{\pm}^{ABEF} P_{\pm}^{EFCD} = P_{\pm}^{ABCD} + \frac{1}{2} n_{AB}^{(+)\dot{0}} n_{CD}^{(+)\dot{0}}, \quad P_{\pm}^{ABEF} P_{\mp}^{EFCD} = -\frac{1}{2} n_{AB}^{(+)\dot{0}} n_{CD}^{(+)\dot{0}} \quad (3.34)$$

by using the fact  $l_{AB}^{(+)\hat{a}} n_{AB}^{(+)\dot{0}} = m_{AB}^{(+)\dot{a}} n_{AB}^{(+)\dot{0}} = 0$ . Of course, Eq. (3.34) is consistent with Eq. (3.29).

Similarly, one can decompose the 't Hooft symbols in Eq. (B.1) into eigenspaces of the operator (3.25):

$$\begin{aligned} l_{AB}^{(-)\hat{a}} &\equiv \left\{ \bar{\eta}_{AB}^{13} = \frac{i}{2} \sigma^2 \otimes \lambda_1, \quad \bar{\eta}_{AB}^{14} = \frac{i}{2} \mathbf{I}_2 \otimes \lambda_2, \quad \frac{1}{\sqrt{3}} (\bar{\eta}_{AB}^8 - \sqrt{2} \bar{\eta}_{AB}^{15}) = -\frac{i}{2} \sigma^2 \otimes \lambda_3, \right. \\ &\quad \bar{\eta}_{AB}^9 = \frac{i}{2} \sigma^2 \otimes \lambda_4, \quad -\bar{\eta}_{AB}^{10} = -\frac{i}{2} \mathbf{I}_2 \otimes \lambda_5, \quad -\bar{\eta}_{AB}^4 = \frac{i}{2} \sigma^2 \otimes \lambda_6, \\ &\quad \left. \bar{\eta}_{AB}^5 = -\frac{i}{2} \mathbf{I}_2 \otimes \lambda_7, \quad -\frac{2}{\sqrt{3}} \left( \frac{1}{2} \bar{\eta}_{AB}^3 + \frac{1}{\sqrt{3}} \bar{\eta}_{AB}^8 + \frac{1}{\sqrt{6}} \bar{\eta}_{AB}^{15} \right) = -\frac{i}{2} \sigma^2 \otimes \lambda_8 \right\} \quad (3.35) \end{aligned}$$

$$\begin{aligned} m_{AB}^{(-)\dot{a}} &\equiv \left\{ \bar{\eta}_{AB}^1 = \frac{i}{2} \sigma^1 \otimes \lambda_2, \quad \bar{\eta}_{AB}^2 = -\frac{i}{2} \sigma^3 \otimes \lambda_2, \quad \bar{\eta}_{AB}^6 = -\frac{i}{2} \sigma^1 \otimes \lambda_5, \right. \\ &\quad \left. -\bar{\eta}_{AB}^7 = \frac{i}{2} \sigma^3 \otimes \lambda_5, \quad -\bar{\eta}_{AB}^{11} = \frac{i}{2} \sigma^1 \otimes \lambda_7, \quad \bar{\eta}_{AB}^{12} = -\frac{i}{2} \sigma^3 \otimes \lambda_7 \right\}, \quad (3.36) \end{aligned}$$

$$n_{AB}^{(-)\dot{0}} \equiv \left\{ \bar{\eta}_{AB}^3 - \frac{1}{\sqrt{3}} \bar{\eta}_{AB}^8 - \frac{1}{\sqrt{6}} \bar{\eta}_{AB}^{15} = \frac{1}{2} \bar{I}_{AB} = \frac{i}{2} \sigma^2 \otimes \mathbf{I}_3 \right\}. \quad (3.37)$$

The exactly same properties such as Eqs. (3.33) and (3.34) hold for the above 't Hooft symbols.

The geometrical meaning of the projection operators in Eq. (3.25) can be understood as follows. Consider an arbitrary two-form

$$F = \frac{1}{2} F_{MN} dx^M \wedge dx^N = \frac{1}{2} F_{AB} E^A \wedge E^B \in \Lambda^2 T^* M \quad (3.38)$$

and introduce the 15-dimensional basis of two-forms in  $\Lambda^2 T^* M$  for each chirality of  $SO(6)$  Lorentz algebra

$$J_{+}^a \equiv \frac{1}{2} \eta_{AB}^a E^A \wedge E^B, \quad J_{-}^a \equiv \frac{1}{2} \bar{\eta}_{AB}^a E^A \wedge E^B. \quad (3.39)$$

It is easy to derive the following useful identity from Eqs. (B.9) and (B.10)

$$J_{\pm}^a \wedge J_{\pm}^b \wedge J_{\pm}^c = \pm \frac{1}{2} d^{abc} \text{vol}(g) \quad (3.40)$$

where  $\text{vol}(g) = \sqrt{g}d^6x$ . The Hodge star is an isomorphism of vector bundle  $*$  :  $\Lambda^k T^*M \rightarrow \Lambda^{6-k} T^*M$  which depends upon a metric  $g$  and the orientation of  $M$ . Consider a nondegenerate 2-form on  $M$

$$\Omega = \frac{1}{2} \bar{I}_{AB} E^A \wedge E^B = E^1 \wedge E^2 + E^3 \wedge E^4 + E^5 \wedge E^6. \quad (3.41)$$

This two-form can be wedged with the Hodge star to construct a diagonalizable operator on  $\Lambda^2 T^*M$  as follows:

$$*_\Omega \equiv *(\bullet \wedge \Omega) : \Lambda^2 T^*M \xrightarrow{\bullet \wedge \Omega} \Lambda^4 T^*M \xrightarrow{*} \Lambda^2 T^*M \quad (3.42)$$

by  $*_\Omega(\alpha) = *(\alpha \wedge \Omega)$  for  $\alpha \in \Lambda^2 T^*M$ . The  $15 \times 15$  matrix representing  $*_\Omega$  is found to have eigenvalues 2, 1 and  $-1$  with eigenspaces of dimension 1, 6 and 8, respectively. On any six-dimensional Riemannian manifold  $M$ , the space of 2-forms  $\Lambda^2 T^*M$  can thus be decomposed into three subspaces

$$\Lambda^2 T^*M = \Lambda_1^2 \oplus \Lambda_6^2 \oplus \Lambda_8^2 \quad (3.43)$$

where  $\Lambda_1^2$  and  $\Lambda_6^2$  are locally spanned by

$$\Lambda_1^2 = \Omega, \quad (3.44)$$

$$\Lambda_6^2 = \{J_+^1, J_+^2, J_+^4, J_+^5, J_+^9, J_+^{10}\}, \quad (3.45)$$

and  $\Lambda_8^2$  by

$$\Lambda_8^2 = \{J_+^6, J_+^7, J_+^{11}, J_+^{12}, J_+^{13}, J_+^{14}, K_+, L_+\}. \quad (3.46)$$

with  $K_+ \equiv \frac{1}{\sqrt{3}}(J_+^8 - \sqrt{2}J_+^{15})$  and  $L_+ \equiv \frac{2}{\sqrt{3}}(-\frac{1}{2}J_+^3 + \frac{1}{\sqrt{3}}J_+^8 + \frac{1}{\sqrt{6}}J_+^{15})$ . A similar decomposition can be done with the negative chirality basis  $J_-^a$ . Later we will explain why there is such a decomposition of the 15-dimensional vector space  $\Lambda^2 T^*M$ .

Note that the entries of  $\Lambda_1^2$ ,  $\Lambda_6^2$  and  $\Lambda_8^2$  coincide with those of  $n_{AB}^{(\pm)0}$ ,  $m_{AB}^{(\pm)\hat{a}}$  and  $l_{AB}^{(\pm)\hat{a}}$ , respectively. One can quickly see that this coincidence is not an accident. Consider the action of the projection operator (3.25) on the two-form (3.38) given by

$$P_\pm^{ABCD} F_{CD} = \frac{1}{2} (F_{AB} \pm \frac{1}{4} \varepsilon^{AB CDEF} F_{CD} \bar{I}_{EF}) \quad (3.47)$$

or in terms of form notation

$$2P_\pm F = F \pm *(F \wedge \Omega) = F \pm *_\Omega F. \quad (3.48)$$

Hence we see that, if  $F \in \Lambda_8^2 = \{F | P_+ F = 0\}$ , it satisfies the  $\Omega$ -anti-self-duality equation

$$F = -*(F \wedge \Omega), \quad (3.49)$$

whereas  $F \in \Lambda_6^2 = \{F | P_- F = 0\}$  satisfies the  $\Omega$ -self-duality equation

$$F = *(F \wedge \Omega). \quad (3.50)$$



It is not difficult to show [2] that the set  $\{l_{AB}^{(\pm)\hat{a}}, n_{AB}^{(\pm)0}\}$  can be identified with  $U(3)$  generators which are embedded in  $SO(6)$ . In general, an element of  $U(3)$  group can be represented as

$$U = \exp \left( i \sum_{a=0}^8 \theta^a \lambda_a \right) \equiv e^\Theta \quad (3.51)$$

where  $\lambda_0 = \mathbf{I}_3$  is a unit matrix of rank 3,  $\lambda_{\hat{a}}$  ( $\hat{a} = 1, \dots, 8$ ) are the  $SU(3)$  Gell-Mann matrices and  $\theta^a$ 's are real parameters for  $U$  to be unitary. Now the  $3 \times 3$  anti-Hermitian matrix  $\Theta$  with matrix elements which are complex numbers  $\Theta_{ij}$  ( $i, j = 1, 2, 3$ ) can easily be embedded into a  $6 \times 6$  real matrix in  $SO(6)$  by replacing  $\Theta_{ij} = \text{Re}\Theta_{ij} + i\text{Im}\Theta_{ij}$  by the  $2 \times 2$  real matrix  $\tilde{\Theta}_{ij} = \mathbf{I}_2 \cdot \text{Re}\Theta_{ij} + i\sigma^2 \cdot \text{Im}\Theta_{ij}$ . A straightforward calculation shows that the resulting  $6 \times 6$  antisymmetric real matrix  $\tilde{\Theta}_{AB}$  can be written as

$$\tilde{\Theta}_{AB} = 2(\theta^0 n_{AB}^{(\pm)0} + \theta^{\hat{a}} l_{AB}^{(\pm)\hat{a}}) = P_-^{ABCD} \tilde{\Theta}_{CD} + 3\theta^0 n_{AB}^{(\pm)0}. \quad (3.52)$$

Note that  $U(3)$  is the holonomy group of a Kähler manifold. That is, the projection operators in Eq. (3.25) can serve to project a Riemannian manifold whose holonomy group is  $SO(6) \cong SU(4)$  to a Kähler manifold with  $U(3)$  holonomy. Now we will show that it is indeed the case. Suppose that  $M$  is a complex manifold and let us introduce local complex coordinates  $z^\alpha = \{x^1 + ix^2, x^3 + ix^4, x^5 + ix^6\}$ ,  $\alpha = 1, 2, 3$  and their complex conjugates  $\bar{z}^{\bar{\alpha}}$ ,  $\bar{\alpha} = 1, 2, 3$ , in which an almost complex structure  $J$  takes the form  $J^\alpha_\beta = i\delta^\alpha_\beta$ ,  $J^{\bar{\alpha}}_{\bar{\beta}} = -i\delta^{\bar{\alpha}}_{\bar{\beta}}$  [2]. Note that, relative to the real basis  $x^M$ ,  $M = 1, \dots, 6$ , the almost complex structure is given by  $J = \bar{I} = i\sigma^2 \otimes \mathbf{I}_3$  which was already introduced in Eq. (3.25). We further impose a Hermitian condition on the complex manifold  $M$  defined by  $g(X, Y) = g(JX, JY)$  for any  $X, Y \in TM$ . This means that a Riemannian metric  $g$  on a complex manifold  $M$  is a Hermitian metric, i.e.,  $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$ ,  $g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}$ . The Hermitian condition can be solved by taking the vielbeins as

$$E_{\bar{\alpha}}^i = E_{\alpha}^{\bar{i}} = 0 \quad \text{or} \quad E_i^{\bar{\alpha}} = E_{\bar{i}}^{\alpha} = 0 \quad (3.53)$$

where a tangent space index  $A = 1, \dots, 6$  has been split into a holomorphic index  $i = 1, 2, 3$  and an anti-holomorphic index  $\bar{i} = 1, 2, 3$ . This in turn means that  $J^i_j = i\delta^i_j$ ,  $J^{\bar{i}}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}}$ . Then one can see that the two-form  $\Omega$  in Eq. (3.41) is a Kähler form, i.e.,  $\Omega(X, Y) = g(JX, Y)$  and it is given by

$$\Omega = iE^i \wedge E^{\bar{i}} = iE_{\alpha}^i E_{\bar{\beta}}^{\bar{i}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}} = ig_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}} \quad (3.54)$$

where  $E^i = E_{\alpha}^i dz^\alpha$  is a holomorphic one-form and  $E^{\bar{i}} = E_{\bar{\alpha}}^{\bar{i}} d\bar{z}^{\bar{\alpha}}$  is an anti-holomorphic one-form. It is then easy to see that the condition that a Hermitian manifold  $(M, g)$  is a Kähler manifold, i.e.  $d\Omega = 0$ , is equivalent to the one that the spin connection  $\omega^A_B$  is  $U(3)$ -valued, i.e.,

$$\omega_{ij} = \omega_{\bar{i}\bar{j}} = 0. \quad (3.55)$$

Therefore, the spin connections after the Kähler condition (3.55) can be written as the form (3.52).

All the above results can be clearly understood by the properties of  $SO(6)$  and  $SU(4)$  groups. Introducing complex coordinates on  $\mathbb{R}^6$  means that the  $\mathbf{4}$  and  $\overline{\mathbf{4}}$  of  $SU(4)$  decompose as  $\mathbf{4} = \mathbf{1}_1 \oplus \mathbf{3}_{-\frac{1}{3}}$  and  $\overline{\mathbf{4}} = \overline{\mathbf{1}}_{-1} \oplus \overline{\mathbf{3}}_{\frac{1}{3}}$  under  $U(1) \times SU(3)$  where the subscripts denote  $U(1)$  charges, because one has to now consider a Lorentz subalgebra  $U(1) \times SU(3) \subset SU(4)$  acting on  $\mathbb{C}^3 \subset \mathbb{C}^4$ . Using the branching rule of  $SU(4) \supset U(1) \times SU(3)$  [10], one can get the following decompositions after removing  $SU(4)$  singlets

$$\mathbf{4} \otimes \overline{\mathbf{4}} - \mathbf{1} = (\mathbf{3} \otimes \overline{\mathbf{3}})_0 \oplus (\mathbf{3}_{-\frac{4}{3}} \oplus \overline{\mathbf{3}}_{\frac{4}{3}}) = (\mathbf{8} \oplus \mathbf{1})_0 \oplus (\mathbf{3}_{-\frac{4}{3}} \oplus \overline{\mathbf{3}}_{\frac{4}{3}}), \quad (3.56)$$

$$\overline{\mathbf{4}} \otimes \mathbf{4} - \mathbf{1} = (\overline{\mathbf{3}} \otimes \mathbf{3})_0 \oplus (\mathbf{3}_{-\frac{4}{3}} \oplus \overline{\mathbf{3}}_{\frac{4}{3}}) = (\mathbf{8} \oplus \mathbf{1})_0 \oplus (\mathbf{3}_{-\frac{4}{3}} \oplus \overline{\mathbf{3}}_{\frac{4}{3}}). \quad (3.57)$$

The spin connection  $\omega_{AB} \in \mathbf{15}$  can be decomposed according to the above branching rule

$$\begin{aligned} \omega_{ij} &\in \mathbf{3}_{-\frac{4}{3}}, & \omega_{\bar{i}\bar{j}} &\in \overline{\mathbf{3}}_{\frac{4}{3}}, \\ \omega_{\bar{i}j} - \frac{1}{3}\delta_j^i \omega_{\bar{k}k} &\in \mathbf{8}_0, & \omega_{\bar{i}i} &\in \mathbf{1}_0. \end{aligned} \quad (3.58)$$

Hence the Kähler condition (3.55) means that spin connections in  $\mathbf{3}_{-\frac{4}{3}}$  and  $\overline{\mathbf{3}}_{\frac{4}{3}}$  decouple from the theory and only the components in  $\mathbf{8}_0$  and  $\mathbf{1}_0$  survive. It is now obvious why we could have such decompositions in Eqs. (3.30)-(3.37) in which  $l_{AB}^{(\pm)\hat{a}} \in \mathbf{8}_0$ ,  $m_{AB}^{(\pm)\hat{a}} \in (\mathbf{3}_{-\frac{4}{3}} \oplus \overline{\mathbf{3}}_{\frac{4}{3}})$  and  $n_{AB}^{(\pm)0} \in \mathbf{1}_0$ .

One can rephrase the Kähler condition (3.55) using the gauge theory formalism. From our identification (2.6), we get the following relation

$$A^{(+)\hat{a}} = -\omega_{AB} \text{Tr}(T^{\hat{a}} J_+^{AB}) = \omega_{AB} \eta_{AB}^{\hat{a}}, \quad \omega_{AB} = A^{(+)\hat{a}} \eta_{AB}^{\hat{a}}, \quad (3.59)$$

$$A^{(-)\hat{a}} = -\omega_{AB} \text{Tr}(T^{\hat{a}} J_-^{AB}) = \omega_{AB} \overline{\eta}_{AB}^{\hat{a}}, \quad \omega_{AB} = A^{(-)\hat{a}} \overline{\eta}_{AB}^{\hat{a}}, \quad (3.60)$$

where we used the definition (3.15). We will focus on the type  $\mathbb{A}$  case in Eq. (3.59) as the same analysis can be applied to the type  $\mathbb{B}$  case (3.60). If  $(M, g)$  is a Kähler manifold, Eq. (3.55) means that

$$A^{(+)\hat{1}} = A^{(+)\hat{2}} = A^{(+)\hat{4}} = A^{(+)\hat{5}} = A^{(+)\hat{9}} = A^{(+)\hat{10}} = 0 \quad (3.61)$$

because  $\eta_{ij}^{\hat{a}} \neq 0$  only for  $a = 1, 2, 4, 5, 9, 10$ , otherwise  $\eta_{ij}^{\hat{a}} = 0$ . See Eq. (B.17). It should also be obvious from the branching rule  $m_{AB}^{(\pm)\hat{a}} \in (\mathbf{3}_{-\frac{4}{3}} \oplus \overline{\mathbf{3}}_{\frac{4}{3}})$ . And then the  $SU(4)$  structure constants  $f^{abc}$  in the Table 1 guarantee that the corresponding field strengths also vanish, i.e.,

$$\begin{aligned} F^{(+)\hat{a}} &= dA^{(+)\hat{a}} - \frac{1}{2} f^{abc} A^{(+)\hat{b}} \wedge A^{(+)\hat{c}} \\ &= \frac{1}{2} f_{(++)}^{ab} \eta_{AB}^{\hat{a}} E^A \wedge E^B = 0 \end{aligned} \quad (3.62)$$

for  $a = 1, 2, 4, 5, 9, 10$ . In other words,  $A^{(+)\hat{a}} = 0$ ,  $f_{(++)}^{\hat{a}b} = 0$  for  $\forall b = 1, \dots, 15$  and so the gauge fields in Eq. (3.59) take values in  $U(3)$  Lie algebra according to the result (3.52). This immediately leads to the conclusion that

$$\begin{aligned} F^{(+)\hat{a}} &= dA^{(+)\hat{a}} - \frac{1}{2} f^{abc} A^{(+)\hat{b}} \wedge A^{(+)\hat{c}} \\ &= f_{(++)}^{ab} J_+^b \in \Lambda_8^2 \oplus \Lambda_1^2 \end{aligned} \quad (3.63)$$

where the field strengths  $F^{(+)\hat{a}}$ ,  $\hat{a} = 0, 1, \dots, 8$  are defined by Eq. (2.9) with  $U(3)$  generators  $T^a$ . As will be shown below,  $F^{(+)\hat{0}} \in \Lambda_1^2$  is the field strength of the  $U(1)$  part of the spin connection on a Kähler manifold and  $F^{(+)\hat{a}} \in \Lambda_8^2$ ,  $\hat{a} = 1, \dots, 8$  belong to the  $SU(3)$  part. In particular, as  $F^{(+)\hat{a}} \in \Lambda_8^2$ , they satisfy the  $\Omega$ -anti-self-duality equation (3.49) known as the Hermitian Yang-Mills equations (or Donaldson-Uhlenbeck-Yau equations) [2]

$$F^{(+)\hat{a}} = - * (F^{(+)\hat{a}} \wedge \Omega), \quad \hat{a} = 1, \dots, 8. \quad (3.64)$$

These results should be expected from the branching rule (3.58).

It is well-known [2] that the Ricci-tensor of a Kähler manifold is the field strength of the  $U(1)$  part of the spin connection. Therefore, the Ricci-flat condition corresponds to the result  $F^{(+)\hat{0}} = 0$ . One can explicitly check it as follows. Recall that  $F_{AB}^{(+)\hat{a}} = f_{(++)}^{ab} \eta_{AB}^{\hat{a}}$ . Using the result (3.61), one can see that the nonzero components of  $f_{(++)}^{ab}$  run only over  $a, b = 3, 6, 7, 8, 11, 12, 13, 14, 15$ . Thereby the constraint (3.24) becomes nontrivial only for those values. As a result, the number of independent components of  $f_{(++)}^{ab}$  is given by  $\frac{9 \times 10}{2} - 9 = 36$ . The Ricci-flat condition  $R_{AB} = f_{(++)}^{ab} \eta_{AC}^a \eta_{BC}^b = 0$  further constrains the coefficients. A close inspection shows that out of 21 equations,  $R_{AB} = 0$ , only 9 equations are independent and, after utilizing the constraint (3.24), the equations for the Ricci-flatness can be succinctly arranged as

$$f_{(++)}^{3a} + \frac{1}{\sqrt{3}} f_{(++)}^{8a} + \frac{1}{\sqrt{6}} f_{(++)}^{15a} = 0. \quad (3.65)$$

The above condition in turn means that

$$\begin{aligned} F_{AB}^{(+)\hat{0}} &= \left( f_{(++)}^{3a} + \frac{1}{\sqrt{3}} f_{(++)}^{8a} + \frac{1}{\sqrt{6}} f_{(++)}^{15a} \right) \eta_{AB}^a \\ &= F_{AB}^{(+)\hat{3}} + \frac{1}{\sqrt{3}} F_{AB}^{(+)\hat{8}} + \frac{1}{\sqrt{6}} F_{AB}^{(+)\hat{15}} = 0. \end{aligned} \quad (3.66)$$

If one introduces a gauge field defined by

$$A^{(+)\hat{0}} \equiv \omega_{AB} n_{AB}^{(+)\hat{0}} = A^{(+)\hat{3}} + \frac{1}{\sqrt{3}} A^{(+)\hat{8}} + \frac{1}{\sqrt{6}} A^{(+)\hat{15}}, \quad (3.67)$$

one can show that the field strength in Eq. (3.66) is given by

$$F^{(+)\hat{0}} = dA^{(+)\hat{0}} \quad (3.68)$$

where it is necessary to use the fact that the  $U(3)$  structure constants  $f^{abc}$  satisfy the following relation

$$f^{3ab} + \frac{1}{\sqrt{3}} f^{8ab} + \frac{1}{\sqrt{6}} f^{15ab} = 0. \quad (3.69)$$

The relation (3.69) is easy to understand because the  $U(1)$  part among the  $U(3)$  structure constants has to vanish. This establishes the result that the Ricci-flatness is equal to the vanishing of the  $U(1)$  field strength. That is,  $F^{(+)\hat{0}} = dA^{(+)\hat{0}} \in \Lambda_1^2$  has a trivial first Chern class.

Again the above result is consistent with the branching rule (3.58). In terms of complex coordinates, the  $U(1)$  gauge field in Eq. (3.67) on a Kähler manifold is given by

$$A^{(+ )0} = i\omega_{\bar{i}i} = -i(E_i^\alpha dE_\alpha^i - E_{\bar{i}}^{\bar{\alpha}} dE_{\bar{\alpha}}^{\bar{i}}) + i(E_i^\alpha \partial_\alpha E^i - E_{\bar{i}}^{\bar{\alpha}} \bar{\partial}_{\bar{\alpha}} E^{\bar{i}}) \in \mathbf{1}_0 \quad (3.70)$$

where the exterior derivative is defined by  $d = \partial + \bar{\partial} = dz^\alpha \partial_\alpha + d\bar{z}^{\bar{\alpha}} \bar{\partial}_{\bar{\alpha}}$ . It is obvious that  $A^{(+ )0}$  cannot be written as an exact one-form, say,  $A^{(+ )0} \neq d\lambda$  with an arbitrary real function  $\lambda(z, \bar{z})$ , though it is closed, i.e.  $dA^{(+ )0} = 0$ . Therefore, one can see that the  $U(1)$  gauge field  $A^{(+ )0} \in \mathbf{1}_0$  is a nontrivial cohomology element.

In summary, the Kähler condition (3.55) projects the 't Hooft symbols to  $U(3)$ -valued ones in  $\mathbf{1}_0 \oplus \mathbf{8}_0$  and results in the reduction of the gauge group from  $SU(4)$  to  $U(3)$ . After imposing the trivial first Chern class,  $F^{(+ )0} = dA^{(+ )0} = 0 \in \mathbf{1}_0$ , the gauge group is further reduced to  $SU(3)$ . Remaining spin connections in  $\mathbf{8}_0$  that are  $SU(3)$  gauge fields satisfy the Hermitian Yang-Mills equation (3.64). As a Kähler manifold with the trivial first Chern class is a Calabi-Yau manifold, this means that the Calabi-Yau manifold is described by the Hermitian Yang-Mills equation (3.64) whose solution is known as Hermitian Yang-Mills instantons [2]. Consequently, we find that six-dimensional Calabi-Yau manifolds are equivalent to Hermitian Yang-Mills instantons in  $SU(3)$  Yang-Mills gauge theory. This equivalence will be more clarified using the gauge theory approach in Section 5.

The same formulae can be obtained for the type  $\mathbb{B}$  case in Eq. (3.60) where the Kähler condition (3.55) is solved by

$$A^{(-)1} = A^{(-)2} = A^{(-)6} = A^{(-)7} = A^{(-)11} = A^{(-)12} = 0. \quad (3.71)$$

The Ricci-flat condition  $R_{AB} = f_{(- -)}^{ab} \bar{\eta}_{AC}^a \bar{\eta}_{BC}^b = 0$  leads to the equation

$$f_{(- -)}^{3a} - \frac{1}{\sqrt{3}} f_{(- -)}^{8a} - \frac{1}{\sqrt{6}} f_{(- -)}^{15a} = 0, \quad (3.72)$$

which is equal to the vanishing of  $U(1)$  field strength, i.e.  $F^{(-)0} = dA^{(-)0} = 0$ , where the  $U(1)$  gauge field is given by

$$A^{(-)0} \equiv \omega_{AB} n_{AB}^{(-)0} = A^{(-)3} - \frac{1}{\sqrt{3}} A^{(-)8} - \frac{1}{\sqrt{6}} A^{(-)15}. \quad (3.73)$$

This can be derived by using the fact that the  $U(3)$  structure constants  $f^{abc}$  also satisfy the following relation

$$f^{3ab} - \frac{1}{\sqrt{3}} f^{8ab} - \frac{1}{\sqrt{6}} f^{15ab} = 0 \quad (3.74)$$

where  $a, b$  now runs over 3, 4, 5, 8, 9, 10, 13, 14, 15. Note that the entries of  $U(3)$  generators for the type  $\mathbb{B}$  case are different from those in the type  $\mathbb{A}$  case. As can be expected, Calabi-Yau manifolds for the type  $\mathbb{B}$  case are also described by the Hermitian Yang-Mills equations

$$F^{(-)\hat{a}} = - * (F^{(-)\hat{a}} \wedge \Omega), \quad \hat{a} = 1, \dots, 8. \quad (3.75)$$

## 4 Mirror Symmetry of Calabi-Yau Manifolds

In this section we want to explore the geometrical properties of six-dimensional Riemannian manifolds in the spinor representations  $\mathbb{A}$  and  $\mathbb{B}$ . For that purpose, consider two generic Riemannian manifolds whose metrics are given by

$$ds_{\mathbb{A}}^2 = E^A \otimes E^A, \quad ds_{\mathbb{B}}^2 = \tilde{E}^A \otimes \tilde{E}^A, \quad (4.1)$$

where  $\mathbb{A}$  and  $\mathbb{B}$  will eventually refer to the chirality classes. Each of the metrics will define their own connections  $\omega^A_B$  through the torsion-free conditions [4],  $T_{\mathbb{A}}^A = dE^A + \omega_{\mathbb{A}B}^A \wedge E^B = 0$  and  $T_{\mathbb{B}}^A = d\tilde{E}^A + \omega_{\mathbb{B}B}^A \wedge \tilde{E}^B = 0$ . The spin connections can take arbitrary values as far as they satisfy the integrability condition (3.20). Their symmetry properties can be characterized by decomposing them into irreducible subspaces under  $SO(6)$  group:

$$\omega_{ABC} \in \mathbf{6} \otimes \mathbf{15} = \square \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = \mathbf{20} \oplus \mathbf{70} \quad (4.2)$$

where  $\begin{array}{|c|} \hline \square \\ \hline \end{array} = \mathbf{20}$  is a completely antisymmetric part of spin connections defined by  $\omega_{[ABC]} = \frac{1}{3}(\omega_{ABC} + \omega_{BCA} + \omega_{CAB})$ . In six dimensions, the spin connections  $\omega_{[ABC]}$  may be further decomposed as (imaginary) self-dual and anti-self-dual parts, i.e.,

$$\omega_{[ABC]} = \left( \omega_{[ABC]}^+ \in \mathbf{10} \right) \oplus \left( \omega_{[ABC]}^- \in \mathbf{10} \right). \quad (4.3)$$

The above decomposition may be shaky because  $\begin{array}{|c|} \hline \square \\ \hline \end{array} = \mathbf{20}$  is already an irreducible representation of  $SO(6)$ . It is just for a heuristic comparison with irreducible  $SU(4)$  representations. Note that  $\mathbf{6}$  is coming from the antisymmetric tensor in  $4 \times 4$  in Eq. (3.8) or  $\bar{4} \times \bar{4}$  in Eq. (3.9). Thus, under  $SU(4)$  group, one can instead get the following decomposition of the spin connections [10]

$$\omega_{ABC} \in \mathbf{6} \otimes \mathbf{15} = \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} = \left( \mathbf{6} = \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \oplus \left( \mathbf{10} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) \oplus \left( \mathbf{\overline{10}} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) \oplus \left( \mathbf{64} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right). \quad (4.4)$$

So notice that the irreducible representation of  $SO(6)$  for spin connections is different from that of  $SU(4)$  which is identified with the irreducible *spinor* representation of  $SO(6)$ .

Introduce a three-form defined by

$$\Psi \equiv \frac{1}{2} \omega_{AB} \wedge E^A \wedge E^B = A^{(\pm)a} \wedge J_{\pm}^a \quad (4.5)$$

where we used the definition in Eqs. (3.39), (3.59) and (3.60). Using the first Bianchi identity (3.20), one can show that it satisfies

$$d\Psi = -dA^{(\pm)a} \wedge J_{\pm}^a = -\frac{1}{2} A^{(\pm)a} \wedge dJ_{\pm}^a. \quad (4.6)$$

One can go further with the identity (4.6). Using the definition  $F^{(\pm)a} = dA^{(\pm)a} - \frac{1}{2} f_{bc}^a A^{(\pm)b} \wedge A^{(\pm)c}$ , the following relation can be derived from Eq. (4.6)

$$\begin{aligned} F^{(\pm)a} \wedge J_{\pm}^a &= \frac{1}{2} A^{(\pm)a} \wedge D_{(\pm)} J_{\pm}^a \\ &= f_{(\pm\pm)}^{ab} J_{\pm}^a \wedge J_{\pm}^b \end{aligned} \quad (4.7)$$

where  $D_{(\pm)}J_{\pm}^a = dJ_{\pm}^a - f_{bc}^a A^{(\pm)b} \wedge J_{\pm}^c$  and the definitions in Eqs. (3.16) and (3.17) are used. By writing  $\omega_{AB} = \omega_{CAB}E^C$ , one can see that

$$\Psi = \frac{1}{2}\omega_{[ABC]}E^A \wedge E^B \wedge E^C \in \mathbf{10} \oplus \overline{\mathbf{10}}. \quad (4.8)$$

Also note that

$$\omega_{ABC} = \frac{1}{2}(f_{ABC} - f_{BCA} + f_{CAB}) \quad (4.9)$$

where  $f_{ABC}$  are structure constants satisfying the Lie algebra (2.15). Therefore, the 3-form  $\Psi$  in Eq. (4.8) can be written as

$$\Psi = \frac{1}{4}f_{ABC}E^A \wedge E^B \wedge E^C = \frac{1}{2}dE^A \wedge E^A \quad (4.10)$$

where we used the structure equation (2.14), which leads to the result

$$d\Psi = \frac{1}{2}dE^A \wedge dE^A = \frac{1}{8}f_{AB}^E f_{CDE}E^A \wedge E^B \wedge E^C \wedge E^D. \quad (4.11)$$

Suppose that  $(M, g)$  is a Kähler manifold, i.e.  $d\Omega = dJ_{\pm}^0 = 0$ . On a Kähler manifold  $M$ , the gauge fields  $A^{(\pm)a}$  take values in  $U(3)$  Lie algebra, namely satisfying Eq. (3.61) or (3.71), and so the three-form  $\Psi$  can be expanded as

$$\Psi = A^{(\pm)0} \wedge \Omega + A^{(\pm)\hat{a}} \wedge J_{\pm}^{\hat{a}}. \quad (4.12)$$

As was shown in Eq. (3.58), this means that the spin connections on the Kähler manifold take values only in  $\mathbf{1}_0$  and  $\mathbf{8}_0$ . Using the branching rule  $\mathbf{6} = \mathbf{3}_{\frac{2}{3}} \oplus \overline{\mathbf{3}}_{-\frac{2}{3}}$  under  $SU(3) \times U(1)$  [10], one can identify the surviving spin connections after the Kähler condition (3.55):

$$\mathbf{6} \otimes \mathbf{15} \rightarrow (\mathbf{3}_{\frac{2}{3}} \oplus \overline{\mathbf{3}}_{-\frac{2}{3}}) \otimes (\mathbf{1}_0 \oplus \mathbf{8}_0). \quad (4.13)$$

Consequently the spin connections on the Kähler manifold  $(M, g)$  can be decomposed as

$$\omega_{ABC} \in (\mathbf{3}_{\frac{2}{3}} \oplus \overline{\mathbf{3}}_{-\frac{2}{3}}) \oplus (\mathbf{3}_{\frac{2}{3}} \otimes \mathbf{8}_0) \oplus (\overline{\mathbf{3}}_{-\frac{2}{3}} \otimes \mathbf{8}_0). \quad (4.14)$$

The tensor products in Eq. (4.14) could be further decomposed according to the branching rule under  $SU(3) \times U(1)$  [10] but it is not necessary for our purpose.

If the three-form  $\Psi$  is defined on a Calabi-Yau manifold  $M$ , our previous result implies that  $\Psi_0 \equiv \frac{4}{3}P_+\Psi = A^{(\pm)0} \wedge \Omega$  is a closed 3-form, i.e.  $d\Psi_0 = 0$ .<sup>3</sup> This means that  $\Psi_0$  is a nontrivial element of the third cohomology group  $H^3(M)$ , because  $A^{(\pm)0}$  cannot be written as an exact one-form as was pointed out in Eq. (3.70) and  $\Omega$  is the Kähler form in the second cohomology group  $H^2(M)$ . It is obvious from our construction that  $\Psi_0 \in (\mathbf{3}_{\frac{2}{3}} \oplus \overline{\mathbf{3}}_{-\frac{2}{3}})$  in Eq. (4.14) is coming from  $\mathbf{10} = \mathbf{1}_2 \oplus \mathbf{3}_{\frac{2}{3}} \oplus \mathbf{6}_{-\frac{2}{3}}$  and its conjugate  $\overline{\mathbf{10}}$  in Eq. (4.8) and it consists of (2,1)- and (1,2)-forms, i.e.,  $\Psi_0 \in H^{2,1}(M) \oplus H^{1,2}(M)$  in the Dolbeault cohomology.

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<sup>3</sup>Note that one cannot use Eq. (4.6) to prove  $d\Psi_0 = 0$  because it is no longer true after the projection  $P_+\Psi$ . That is, it is necessary to have both  $dA^{(\pm)0} = 0$  and  $d\Omega = 0$  to verify the closedness.

Now let us summarize where nontrivial classes of Dolbeault cohomology on a Calabi-Yau manifold come from. We observed that a part of them is coming from the metrics in (4.1) and the other is coming from the spin connections in (4.4). It is well-known [2] that a compact Kähler manifold has a nontrivial second cohomology group  $H^2(M)$  which gives rise to a positive Betti number  $b_2 > 0$ . This second cohomology  $H^2(M)$  is coming from the Kähler class  $\Omega$  of a Kähler metric defined by Eq. (3.54). Another nontrivial cohomology class coming from the metric on a Calabi-Yau 3-fold is the holomorphic 3-form  $\Phi$ . A Calabi-Yau 3-fold always admits a globally defined and nowhere vanishing holomorphic volume-form  $\Phi$  satisfying the property [3]

$$\Omega \wedge \Omega \wedge \Omega = i\Phi \wedge \bar{\Phi}. \quad (4.15)$$

Hence the holomorphic 3-form  $\Phi$  is basically defined by the metrics in (4.1) and form a one-dimensional vector bundle called the canonical line bundle  $L$  of  $M$ . And we have learned that the remaining cohomology class is coming from the spin connection  $\Psi_0 \in (\mathbf{3}_{\frac{2}{3}} \oplus \bar{\mathbf{3}}_{-\frac{2}{3}})$  in Eq. (4.12), which belongs to the third cohomology group  $H^3(M)$ , to be specific, to the Dolbeault cohomology  $H^{2,1}(M) \oplus H^{1,2}(M)$ .

The nontrivial cohomology class on a Calabi-Yau manifold  $M$  may be represented as

$$\Omega = iE^i \wedge E^{\bar{i}} \in H^{1,1}(M), \quad (4.16)$$

$$\Phi = E^i \wedge E^j \wedge E^k \in H^{3,0}(M), \quad \bar{\Phi} = E^{\bar{i}} \wedge E^{\bar{j}} \wedge E^{\bar{k}} \in H^{0,3}(M), \quad (4.17)$$

$$\Psi_0^\pm = A_i^{(\pm)0} E^i \wedge \Omega \in H^{2,1}(M), \quad \bar{\Psi}_0^\pm = \bar{A}_{\bar{i}}^{(\pm)0} E^{\bar{i}} \wedge \Omega \in H^{1,2}(M), \quad (4.18)$$

where  $E^i = E_\alpha^i dz^\alpha$  is a holomorphic one-form and  $E^{\bar{i}} = E_{\bar{\alpha}}^{\bar{i}} d\bar{z}^{\bar{\alpha}}$  is an anti-holomorphic one-form. According to the correspondence (3.2), we make the following identification [2]

$$E^i \leftrightarrow a^i, \quad E^{\bar{i}} \leftrightarrow a_i^*, \quad i = 1, 2, 3, \quad (4.19)$$

where  $a^i$  and  $a_i^*$  are annihilation and creation operators, respectively, acting on the Hilbert space  $V$  in Eq. (3.3). Therefore, we identify the cohomology classes in Eqs. (4.16)-(4.18) with the following fermion operators with a symmetric ordering prescription

$$\widehat{\Omega} = \frac{1}{2}(a_i^* a^i - a^i a_i^*) = a_i^* a^i - \frac{3}{2}, \quad (4.20)$$

$$\widehat{\Phi} = a^i a^j a^k, \quad \widehat{\bar{\Phi}} = -a_i^* a_j^* a_k^*, \quad (4.21)$$

$$\begin{aligned} \widehat{\Psi}_0^\pm &= \frac{1}{2} A_i^{(\pm)0} (a^i \widehat{\Omega} + \widehat{\Omega} a^i) = (a_j^* a^j - 1) (A_i^{(\pm)0} a^i), \\ \widehat{\bar{\Psi}}_0^\pm &= \frac{1}{2} \bar{A}^{(\pm)0i} (\widehat{\Omega} a_i^* + a_i^* \widehat{\Omega}) = (a_j^* a^j - 2) (\bar{A}^{(\pm)0i} a_i^*), \end{aligned} \quad (4.22)$$

where the factor  $i$  was absorbed in the definition such that  $\widehat{\Omega}$  becomes a Hermitian operator.

Given a metric  $ds^2 = E^A \otimes E^A$ , one can determine the spin connection  $\omega_{AB}$  via the torsion free condition,  $T^A = dE^A + \omega^A_B \wedge E^B = 0$ . Because we are taking an irreducible spinor representation of  $SO(6)$  for the identification (2.6), it is necessary to specify which representation is chosen to



embed the spin connection  $\omega_{AB}$ . As we remarked in the footnote 2, one can equally choose either the positive chirality representation or the negative chirality representation. Or one may consider the case where a Dirac operator  $D_M = \partial_M + \omega_M$  acts on a chiral spinor  $\eta$  which obeys a well-known commutation relation  $[D_M, D_N]\eta = R_{MNPQ}\Gamma^{PQ}\eta$ . In this case the representation has been fixed from the outset by the chiral spinor  $\eta$ . At any rate, the identification of  $SU(4)$  gauge fields, according to the definition (3.59) and (3.60), depends on which representation has been chosen. When a specific chirality representation is chosen for a given metric, one can determine the coefficients  $f_{(++)}^{ab}$  in Eq. (3.18) or  $f_{(--) }^{ab}$  in Eq. (3.19) according to the chosen representation. Therefore, when one considers two kinds of metrics like Eq. (4.1) which are embedded into different chiral representations, it will be possible to find a pair of metrics such that  $f_{(++)}^{ab} = -f_{(--) }^{ab}$  for all  $a, b$ . However, the Riemannian manifold described by each metric in the pair will not be the same (even topologically) because the curvature tensors given by Eqs. (3.18) and (3.19) are different each other.

So suppose that the spin connections determined by the metrics in Eq. (4.1) are embedded into different chiral representations, as indicated there, so as to obey the relation  $f_{(++)}^{ab} = -f_{(--) }^{ab}$ . For example, one Calabi-Yau manifold  $M$  with the metric given by  $ds_{\mathbb{A}}^2 = E^A \otimes E^A$  is embedded in the spinor representation of the type  $\mathbb{A}$  in Eq. (3.11) while the other Calabi-Yau manifold  $\widetilde{M}$  with the metric  $ds_{\mathbb{B}}^2 = \widetilde{E}^A \otimes \widetilde{E}^A$  is in the spinor representation of the type  $\mathbb{B}$  in Eq. (3.12). Then the two Calabi-Yau manifolds  $M$  and  $\widetilde{M}$  will have their own cohomology classes given by Eqs. (4.16)-(4.18) with  $U(1)$  gauge fields  $A_i^{(+ )0}, \bar{A}^{(+ )0i}$  and  $A_i^{(- )0}, \bar{A}^{(- )0i}$ , respectively. Therefore, let us consider the cohomology operators in Eqs. (4.20)-(4.22) for each Calabi-Yau manifold acting on the Hilbert space  $S_{\pm}$  in Eqs. (3.4) and (3.5). First note that  $\widehat{\Phi}S_+ = 0, \widehat{\Psi}_0^+S_+ = 0$  for the Calabi-Yau manifold  $M$  and  $\widehat{\Phi}S_- = 0, \widehat{\Psi}_0^-S_- = 0$  for another Calabi-Yau manifold  $\widetilde{M}$ . And the Kähler operator  $\widehat{\Omega}$  in (4.20) assigns the eigenvalue  $(-\frac{3}{2}, \frac{1}{2})$  for the positive-chirality state  $S_+ = 4$  in (3.4) and  $(-\frac{1}{2}, \frac{3}{2})$  for the negative-chirality state  $S_- = \bar{4}$  in (3.5). Finally the operators  $\widehat{\Psi}_0^-$  and  $\widehat{\Psi}_0^+$  in (4.22) nontrivially act on  $S_-$  and  $S_+$ , respectively, as follows

$$\widehat{\Psi}_0^-S_- = (a_j^*a^j - 1)\left[(A_i^{(- )0}a^i)S_-\right] = (\widehat{\Omega} + \frac{1}{2})S_+, \quad (4.23)$$

$$\widehat{\Psi}_0^+S_+ = (a_j^*a^j - 2)\left[(\bar{A}^{(+ )0i}a_i^*)S_+\right] = (\widehat{\Omega} - \frac{1}{2})S_-, \quad (4.24)$$

where we used the fact that the creation and annihilation operators change the chirality, i.e.  $(a_i^*, a^i) : S_{\pm} \rightarrow S_{\mp}$ . It might be remarked that the  $\pm\frac{1}{2}$  shift in the Kähler operator  $(\widehat{\Omega} \pm \frac{1}{2})$  is to correct the  $U(1)$  charge difference between  $\widehat{\Omega}$  and  $\widehat{\Psi}_0^{\pm}$  operators. We will see that the relation in Eqs. (4.23) and (4.24) implies an important picture about the mirror symmetry of Calabi-Yau manifolds.

Let us briefly summarize the mirror symmetry of Calabi-Yau manifolds [3]. Let  $M$  be a compact Calabi-Yau manifold. The only non-trivial cohomology of a Calabi-Yau manifold is contained in  $H^{1,1}(M)$  and  $H^{2,1}(M)$  besides the one-dimensional cohomologies  $H^{0,0}(M) = H^{3,3}(M) = H^{3,0}(M) = H^{0,3}(M) = 1$ . These cohomology classes parameterize their moduli. It is known [2] that every  $H^{1,1}(M)$  is represented by a real closed  $(1, 1)$ -form which forms a Kähler class represented by the

Kähler form  $\Omega$  in Eq. (4.16). The elements in  $H^{1,1}(M)$  can infinitesimally change the Kähler structure of a Calabi-Yau manifold and are therefore called Kähler moduli. (In string theory these moduli are usually complexified.) On the other hand,  $H^{2,1}(M)$  parameterizes the complex structure moduli of a Calabi-Yau manifold. It is thanks to the fact that the cohomology class of (2,1)-forms is isomorphic to the cohomology class  $H^1_{\bar{\partial}}(T)$ , the first Dolbeault cohomology group of  $M$  with values in a holomorphic tangent bundle  $T$ , that characterizes infinitesimal complex structure deformations [2].

The mirror symmetry of Calabi-Yau manifolds is the duality between two different Calabi-Yau 3-folds  $M$  and  $\widetilde{M}$  such that the Hodge numbers of  $M$  and  $\widetilde{M}$  satisfy the relations [3]

$$h^{1,1}(M) = h^{2,1}(\widetilde{M}), \quad h^{1,1}(\widetilde{M}) = h^{2,1}(M), \quad (4.25)$$

or in a more general form

$$h^{p,q}(M) = h^{3-p,q}(\widetilde{M}), \quad (4.26)$$

where  $h^{p,q} = \dim H^{p,q}(M)$  is a Hodge number of the Calabi-Yau manifold  $M$  satisfying  $h^{p,q} = h^{q,p}$  and  $h^{p,q} = h^{3-p,3-q}$  [2]. We know that  $h^{1,1} = \dim H^{1,1}(M)$  is the number of possible (in general, complexified) Kähler forms and  $h^{2,1} = \dim H^{2,1}(M)$  is the dimension of the complex moduli of  $M$ . Hence, as we mentioned before, the only non-trivial deformations of a Calabi-Yau manifold are coming from the cohomology classes in  $H^{1,1}(M)$  and  $H^{2,1}(M)$ .

Mirror symmetry suggests that for each Calabi-Yau 3-fold  $M$  there exists another Calabi-Yau 3-fold  $\widetilde{M}$  whose Hodge numbers satisfy the relation (4.25). Recall that the Calabi-Yau manifold  $M$  is of type  $\mathbb{A}$  whose Riemann curvature tensor is given by Eq. (3.18) while the Calabi-Yau manifold  $\widetilde{M}$  is of type  $\mathbb{B}$  whose Riemann curvature tensor is given by Eq. (3.19). In other words, the spinor representation of the Calabi-Yau manifold  $M$  is given by the positive-chirality space  $S_+ = 4$  whereas the other Calabi-Yau manifold  $\widetilde{M}$  is represented in the negative-chirality spinor space  $S_- = \overline{4}$ . And, for each Calabi-Yau manifold, we have identified their cohomology classes in Eqs. (4.16)-(4.18) with the fermion operators in Eqs. (4.20)-(4.22). If one stares at the relation (4.23) and (4.24) with this understanding, one can find exactly the same statement with the mirror symmetry:

$$\widehat{\Psi}_0^{\mathbb{B}} S_- = (\widehat{\Omega}^{\mathbb{A}} + \frac{1}{2}) S_+, \quad \widehat{\Psi}_0^{\mathbb{A}} S_+ = (\widehat{\Omega}^{\mathbb{B}} - \frac{1}{2}) S_-. \quad (4.27)$$

Specifically, the relation (4.27) implies that the Calabi-Yau manifold  $M$  of type  $\mathbb{A}$  is mirror to the Calabi-Yau manifold  $\widetilde{M}$  of type  $\mathbb{B}$ . According to our construction, they belong to different chiral representations.

Finally we will provide an evidence that a pair of Calabi-Yau manifolds are a mirror pair obeying the relation (4.25) if their curvature tensors defined by Eqs. (3.18) and (3.19) satisfy the relation  $f_{(++)}^{ab} = -f_{(--)}^{ab}$ . First of all, we will calculate their Euler characteristics  $\chi(M)$  and  $\chi(\widetilde{M})$  and show that  $\chi(M) = -\chi(\widetilde{M})$ .

Every complex vector bundle  $E$  of rank  $n$  has an underlying real vector bundle  $E_{\mathbb{R}}$  of rank  $2n$ , obtained by discarding the complex structure on each fiber. Then the top Chern class of a complex

vector bundle  $E$  is the Euler class of its realization [11]

$$c_n(E) = e(E_{\mathbb{R}}) \quad (4.28)$$

where  $n = \text{rank } E$ . Therefore, the Euler characteristic  $\chi(M)$  of  $M$  for a tangent bundle  $E_{\mathbb{R}} = TM$  is given by the integral of the top Chern class

$$\chi(M) = \int_M c_n(E). \quad (4.29)$$

Recall that if  $E$  is a complex vector bundle, then there exists a dual or conjugate bundle  $\overline{E}$  with an opposite complex structure whose  $j$ -th Chern class is given by [7, 11]

$$c_j(\overline{E}) = (-1)^j c_j(E). \quad (4.30)$$

The Euler characteristic  $\chi(M)$  for a six-dimensional Riemannian manifold  $M$  is given by

$$\begin{aligned} \chi(M) &\equiv -\frac{1}{2^6 \cdot 6\pi^3} \int_M \varepsilon^{A_1 A_2 \dots A_6} R_{A_1 A_2} \wedge R_{A_3 A_4} \wedge R_{A_5 A_6} \\ &= -\frac{1}{2^9 \cdot 6\pi^3} \int_M d^6 x \varepsilon^{M_1 M_2 \dots M_6} \varepsilon^{A_1 A_2 \dots A_6} R_{M_1 M_2 A_1 A_2} R_{M_3 M_4 A_3 A_4} R_{M_5 M_6 A_5 A_6}. \end{aligned} \quad (4.31)$$

On one hand, for the type  $\mathbb{A}$  in Eq. (3.13) where  $R_{AB} = F^{(+a)} \eta_{AB}^a$ , it is given by

$$\begin{aligned} \chi(M) &= -\frac{1}{2^6 \cdot 6\pi^3} \int_M \varepsilon^{A_1 A_2 \dots A_6} R_{A_1 A_2} \wedge R_{A_3 A_4} \wedge R_{A_5 A_6} \\ &= -\frac{1}{2^6 \cdot 6\pi^3} \int_M (\varepsilon^{A_1 A_2 \dots A_6} \eta_{A_1 A_2}^a \eta_{A_3 A_4}^b \eta_{A_5 A_6}^c) F^{(+a)} \wedge F^{(+b)} \wedge F^{(+c)} \\ &= -\frac{1}{96\pi^3} \int_M d^{abc} F^{(+a)} \wedge F^{(+b)} \wedge F^{(+c)} \end{aligned} \quad (4.32)$$

where Eq. (B.9) was used. On the other hand, for the type  $\mathbb{B}$  in Eq. (3.14) where  $R_{AB} = F^{(-a)} \overline{\eta}_{AB}^a$ , the Euler characteristic in Eq. (4.31) can be written as

$$\begin{aligned} \chi(\widetilde{M}) &= -\frac{1}{2^6 \cdot 6\pi^3} \int_{\widetilde{M}} \varepsilon^{A_1 A_2 \dots A_6} R_{A_1 A_2} \wedge R_{A_3 A_4} \wedge R_{A_5 A_6} \\ &= \frac{1}{96\pi^3} \int_{\widetilde{M}} d^{abc} F^{(-a)} \wedge F^{(-b)} \wedge F^{(-c)} \end{aligned} \quad (4.33)$$

where Eq. (B.10) was used.

Note that two irreducible spinor representations of  $SO(6)$  can be identified with the fundamental and anti-fundamental representations of  $SU(4)$ . By choosing a complex structure, e.g., Eq. (3.53), the  $SO(6)$  tangent bundle  $TM$  reduces to a  $U(3)$  vector bundle  $E$ . In order to utilize the relation (4.29), let us consider a  $U(3) \subset SU(4)$  sub-bundle  $E$  such that  $TM \otimes \mathbb{C} = E \oplus \overline{E}$ . Note that the  $U(3) = SU(3) \times U(1)$  subgroup among the Lorentz group  $SO(6)$  does not mix the creation and

annihilation operators in the Hilbert space (3.3). Thus we will consider the class  $\mathbb{A}$  in Eq. (3.11) as a  $U(3)$  vector bundle  $E$  and the class  $\mathbb{B}$  in Eq. (3.12) as its dual (conjugate) bundle  $\overline{E}$ . According to the definition (4.29), one can see that the sign difference (4.30) between a vector bundle  $E$  and its conjugate bundle  $\overline{E}$  originates from different chiral representations in our case. That is, the sign difference (4.30) for the third Chern class of  $U(3)$  vector bundle  $E$  and its dual bundle  $\overline{E}$  has been attributed to the opposite chirality (or parity).

It is straightforward to represent the Euler characteristics in Eqs. (4.32) and (4.33) in terms of the chiral bases in Eqs. (3.18) and (3.19). For the type  $\mathbb{A}$  in Eq. (3.16) where  $F^{(+a)} = f_{(++)}^{ab} J_+^b$ , one can represent  $\chi(M)$  using the identity in Eq. (3.40) with

$$\begin{aligned}\chi(M) &= -\frac{1}{96\pi^3} \int_M d^{abc} F^{(+a)} \wedge F^{(+b)} \wedge F^{(+c)} \\ &= -\frac{1}{192\pi^3} \int_M d^6x \sqrt{g} d^{abc} d^{def} f_{(++)}^{ad} f_{(++)}^{be} f_{(++)}^{cf},\end{aligned}\quad (4.34)$$

whereas, using the identity in Eq. (3.40) again,  $\chi(\widetilde{M})$  for the type  $\mathbb{B}$  in Eq. (3.17) where  $F^{(-a)} = f_{(--) }^{ab} J_-^b$  can be written as

$$\begin{aligned}\chi(\widetilde{M}) &= \frac{1}{96\pi^3} \int_{\widetilde{M}} d^{abc} F^{(-a)} \wedge F^{(-b)} \wedge F^{(-c)} \\ &= -\frac{1}{192\pi^3} \int_{\widetilde{M}} d^6x \sqrt{\widetilde{g}} d^{abc} d^{def} f_{(--) }^{ad} f_{(--) }^{be} f_{(--) }^{cf}.\end{aligned}\quad (4.35)$$

One can see that the Euler characteristics (4.34) for the type  $\mathbb{A}$  and (4.35) for the type  $\mathbb{B}$  have a precisely opposite sign, i.e.,  $\chi(M) = -\chi(\widetilde{M})$ , if  $f_{(++)}^{ab} = -f_{(--) }^{ab}$  and  $\text{vol}(g) = \text{vol}(\widetilde{g}) = \sqrt{g} d^6x$ . The latter condition can be achieved if  $h^{3,0}(M) = h^{0,0}(\widetilde{M})$  and  $h^{0,3}(M) = h^{3,3}(\widetilde{M})$ . See Eq. (4.26). As was shown in Section 3, for a Kähler manifold, the gauge (holonomy) group reduces to  $U(3)$  and so the structure constant  $d^{abc}$  takes values only in the  $U(3) \subset SU(4)$  Lie algebra. For a Calabi-Yau manifold, the structure constants  $d^{abc}$  now take values only in the  $SU(3) \subset U(3)$  Lie algebra and the Euler characteristic  $\chi(M)$  of a Calabi-Yau manifold  $M$  whose holonomy is precisely  $SU(3)$  is given by [2]

$$\chi(M) = 2(h^{1,1} - h^{2,1}). \quad (4.36)$$

Therefore, the opposite sign between  $\chi(M)$  and  $\chi(\widetilde{M})$  implies the consistent result with the mirror symmetry (4.25), i.e.,

$$h^{1,1}(\mathbb{A}) = h^{2,1}(\mathbb{B}), \quad h^{1,1}(\mathbb{B}) = h^{2,1}(\mathbb{A}). \quad (4.37)$$

Note that the opposite sign between the Euler characteristics  $\chi(M)$  and  $\chi(\widetilde{M})$  cannot come from the sign flip of Hodge numbers, namely  $h^{1,1}(\mathbb{A}) = -h^{1,1}(\mathbb{B})$  and  $h^{2,1}(\mathbb{A}) = -h^{2,1}(\mathbb{B})$ , because of the definition of the Hodge number  $h^{p,q} = \dim H^{p,q}(M) \geq 0$ .

The mirror symmetry (4.37) can be further clarified by using the fact that the Euler characteristic  $\chi(M)$  of a spin manifold  $M$  is related to the *index* of the Dirac operator on  $M$  [12]. Denote the Dirac

index for fermion fields in a representation  $R$  by  $index(R)$ . The Euler characteristic  $\chi(M)$  is then given by

$$\chi(M) = index(R) - index(\overline{R}) \quad (4.38)$$

where  $\overline{R}$  is the complex conjugate of  $R$ . Let  $\mathbf{4}$  be the fundamental representation of  $SU(4)$  and  $\overline{\mathbf{4}}$  its complex conjugate, i.e. the anti-fundamental representation of  $SU(4)$ . Then  $index(\overline{\mathbf{4}}) = -index(\mathbf{4})$  since in six dimensions complex conjugation exchanges positive and negative chirality zero modes while also exchanging  $\mathbf{4}$  and  $\overline{\mathbf{4}}$ . As  $SU(3)$  representations,  $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$  and  $\overline{\mathbf{4}} = \overline{\mathbf{1}} \oplus \overline{\mathbf{3}}$  where  $index(\mathbf{1}) = index(\overline{\mathbf{1}}) = 0$  and so the Euler characteristic (4.38) is given by [12]

$$\begin{aligned} \chi(M) &= index(\mathbf{4}) - index(\overline{\mathbf{4}}) \\ &= 2 index(\mathbf{4}) \\ &= index(\mathbf{3}) - index(\overline{\mathbf{3}}) \\ &= 2 index(\mathbf{3}). \end{aligned} \quad (4.39)$$

Consequently, the identity (4.39) immediately implies the relation  $\chi(M) = -\chi(\widetilde{M})$  for a pair of 6-manifolds embedded in the opposite chirality representations  $\mathbf{4}$  and  $\overline{\mathbf{4}}$  (or  $\mathbf{3}$  and  $\overline{\mathbf{3}}$  for Calabi-Yau manifolds). This result is consistent with the mirror symmetry (4.37) since  $\mathbb{A} \in \mathbf{3}$  and  $\mathbb{B} \in \overline{\mathbf{3}}$ .

## 5 Mirror Symmetry from Gauge Theory

We showed in Section 3 that Calabi-Yau manifolds can be identified with Hermitian Yang-Mills instantons from the gauge theory point of view. And the mirror symmetry says that a Calabi-Yau manifold has a mirror pair satisfying the relation (4.25). Therefore, there must be a corresponding Hermitian Yang-Mills instanton mirror to the original Hermitian Yang-Mills instanton that obeys the mirror property (4.37). So an interesting question is what is the mirror Hermitian Yang-Mills instanton from the viewpoint of gauge theory formulation. This question is the subject of this section.

In Section 3, the six-dimensional Euclidean gravity has been formulated as  $SU(4) \cong SO(6)$  Yang-Mills gauge theory. We found that a Kähler manifold is described by  $U(3)$  gauge theory. After imposing the Ricci-flat condition on the Kähler manifold, the gauge group in the Yang-Mills theory is further reduced to  $SU(3)$  and so a Calabi-Yau manifold is described by  $SU(3)$  Yang-Mills gauge theory. In particular, we found that six-dimensional Calabi-Yau manifolds are equivalent to Hermitian Yang-Mills instantons in the  $SU(3)$  gauge theory. Now we will derive these results again and then discuss the mirror symmetry between Calabi-Yau manifolds from a completely gauge theory setup.

Suppose that the metric of a six-dimensional Riemannian manifold  $M$  is given by

$$ds^2 = g_{MN}(x) dx^M dx^N. \quad (5.1)$$

Let  $\pi : E \rightarrow M$  be an  $SU(4)$  bundle over  $M$  whose curvature is defined by

$$\begin{aligned} F &= dA + A \wedge A = \frac{1}{2} F_{MN}(x) dx^M \wedge dx^N \\ &= \frac{1}{2} \left( \partial_M A_N - \partial_N A_M + [A_M, A_N] \right) dx^M \wedge dx^N \end{aligned} \quad (5.2)$$

where  $A = A_M^a(x) T^a dx^M$  is a connection one-form of the vector bundle  $E$ . The generators  $T^a$  of  $SU(4)$  Lie algebra satisfy the commutation relation (2.5) with a normalization  $\text{Tr} T^a T^b = -\frac{1}{2} \delta^{ab}$ .

Consider  $SU(4)$  Yang-Mills theory defined on the Riemannian manifold (5.1) whose action is given by

$$S_{YM} = -\frac{1}{2g_{YM}^2} \int_M d^6x \sqrt{g} g^{MP} g^{NQ} \text{Tr} F_{MN} F_{PQ}. \quad (5.3)$$

Using the projection operator (3.25) and the identity (3.29), it is easy to derive the following formula

$$\begin{aligned} (P_{\pm} F)^2 &= (P_{\pm}^{A_1 B_1 A_2 B_2} F_{A_2 B_2}) (P_{\pm}^{A_1 B_1 A_3 B_3} F_{A_3 B_3}) \\ &= \frac{1}{4} \left( F_{A_1 B_1} \pm \frac{1}{4} \varepsilon^{A_1 B_1 A_2 B_2 A_3 B_3} F_{A_2 B_2} \bar{I}_{A_3 B_3} \right)^2 \\ &= P_{\pm}^{ABCD} F_{AB} F_{CD} + \frac{1}{8} (\bar{I}_{AB} F^{AB})^2 \\ &= \frac{1}{2} F_{AB} F^{AB} \pm \frac{1}{8} \varepsilon^{ABCDEF} F_{AB} F_{CD} \bar{I}_{EF} + \frac{1}{8} (\bar{I}_{AB} F^{AB})^2. \end{aligned} \quad (5.4)$$

One can rewrite the action (5.3) using the above identity as

$$\begin{aligned} S_{YM} = -\frac{1}{4g_{YM}^2} \int_M d^6x \sqrt{g} \text{Tr} \left[ \left( F_{A_1 B_1} \pm \frac{1}{4} \varepsilon^{A_1 B_1 A_2 B_2 A_3 B_3} F_{A_2 B_2} \bar{I}_{A_3 B_3} \right)^2 - \frac{1}{2} (\bar{I}_{AB} F^{AB})^2 \right. \\ \left. \mp \frac{1}{2} \varepsilon^{ABCDEF} F_{AB} F_{CD} \bar{I}_{EF} \right]. \end{aligned} \quad (5.5)$$

The above action can be written in a more compact form as

$$\begin{aligned} S_{YM} &= -\frac{1}{4g_{YM}^2} \int_M d^6x \sqrt{g} \text{Tr} \left[ \left( F_{AB} \pm *(F \wedge \Omega)_{AB} \right)^2 - \frac{1}{2} (\bar{I}_{AB} F^{AB})^2 \right] \\ &\quad \pm \frac{1}{g_{YM}^2} \int_M d^6x \text{Tr} F \wedge F \wedge \Omega \end{aligned} \quad (5.6)$$

where  $\Omega$  is the two-form of rank 6 defined in Eq. (3.41).

Using the fact

$$\text{Tr} F \wedge F = d \text{Tr} \left( A \wedge F - \frac{1}{3} A \wedge A \wedge A \right) \equiv dK, \quad (5.7)$$

one can see that the last term in Eq. (5.6) is a topological term, i.e.,

$$\text{Tr} F \wedge F \wedge \Omega = d(K \wedge \Omega) \quad (5.8)$$

if and only if the two-form  $\Omega$  is closed, i.e.  $d\Omega = 0$ . In other words, when  $M$  is a Kähler manifold, the last term in Eq. (5.6) depends only on the topological class of the Kähler-form  $\Omega$  and the vector bundle  $E$  on  $M$ . Although we have originally started with the positive definite action (5.3), the term in the brace  $[\dots]$  in Eq. (5.6) after separating a topological term is not necessarily positive definite due to the second term. In order to keep the positive definiteness of the Yang-Mills action (5.6) after separating the topological term, it is necessary to impose the following requirement

$$\bar{I}_{AB}F^{AB} = 0. \quad (5.9)$$

We will see later the geometrical meaning and the significance of the condition (5.9). After the condition (5.9), the action (5.6) is now positive definite up to a topological term and the minimum action can be achieved in a configuration satisfying the equations

$$F_{AB} \pm *(F \wedge \Omega)_{AB} = 0. \quad (5.10)$$

Note that we have already encountered the above equations in Eqs. (3.49) and (3.50). The above  $\Omega$ -self-duality equations together with Eq. (5.9) implies the so-called Donaldson-Uhlenbeck-Yau equation

$$F \wedge \Omega^2 = 0. \quad (5.11)$$

Then the topological term in Eq. (5.6) becomes equal to the original action (5.3) when Eq. (5.10) is satisfied, which explains the reason why we need the condition (5.9). It is easy to see [2, 13] that, if Eqs. (5.9) and (5.10) are satisfied, Yang-Mills equations on a Kähler manifold given by

$$g^{MN}D_M F_{NP} = 0, \quad (5.12)$$

are automatically satisfied.

We want to solve Eqs. (5.9) and (5.10) known as the Hermitian Yang-Mills equations [2]. It was observed in Section 3 that the 't Hooft symbols in Eq. (3.15) realizes the isomorphism between  $SO(6)$  Lorentz algebra and  $SU(4)$  Lie algebra and provide a complete basis of two-forms in  $\Lambda^2 T^*M$ . Thus one may expand the  $SU(4)$  field strength  $F_{AB}^a$  ( $a = 1, \dots, 15$ ) using the basis in Eq. (3.15), for example, like either Eq. (3.16) or Eq. (3.17). But we know that there are two independent bases,  $\eta_{AB}^a$  and  $\bar{\eta}_{AB}^a$ . So the question is how to distinguish the two bases from the  $SU(4)$  gauge theory approach. The crux is that the  $N$ -dimensional fundamental representation of  $SU(N)$  for  $N$  greater than two is a complex representation whose complex conjugate is often called the anti-fundamental representation. And the complex conjugate  $\bar{\mathbf{N}}$  of a complex representation  $\mathbf{N}$  is a different, non-equivalent representation. In particular, the positive and negative chirality representations of  $SO(6) \cong SU(4)$  are four dimensional which coincides with the fundamental ( $\mathbf{4}$ ) and the anti-fundamental ( $\bar{\mathbf{4}}$ ) representations of  $SU(4)$ .<sup>4</sup> This is yet another reason why there are two independent bases of two-forms.

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<sup>4</sup>One may notice that the complex conjugation acting on  $\mathbb{C}^4$  plays a role of the Hodge  $*$ -operator acting on the vector space  $\Lambda^*M = \bigoplus_k \Lambda^k T^*M$  as well as the chiral operator  $\Gamma^7$  in the Clifford algebra  $\mathbb{Cl}(6)$ .



In the anti-fundamental representation  $\bar{4}$  of  $SU(4)$ , the generators may be given by  $(T^a)^* = -\frac{i}{2}\lambda_a^*$  and they obey the same Lie algebra as  $T^a$ :

$$[(T^a)^*, (T^b)^*] = -f^{abc}(T^c)^*. \quad (5.13)$$

But one can see from Eq. (A.9) that the symmetric structure constants  $d^{abc}$  have an opposite sign, i.e.,

$$\text{Tr} \{T^a, T^b\} T^c = -\frac{i}{2}d^{abc}, \quad \text{Tr} \{(T^a)^*, (T^b)^*\} (T^c)^* = \frac{i}{2}d^{abc}. \quad (5.14)$$

It turns out that this sign flip is correlated with the opposite sign in Eq. (3.40).

Thereby we will expand the  $SU(4)$  field strength  $F_{AB}^a$  in the fundamental representation  $4$  as the type  $\mathbb{A}$  in Eq. (3.16) and the type  $\mathbb{B}$  in Eq. (3.17) for the anti-fundamental representation  $\bar{4}$ . According to the tensor product (3.21), one can decompose the coefficients  $f_{(\pm\pm)}^{ab}$  into a symmetric part and an antisymmetric part

$$f_{(\pm\pm)}^{ab} = f_{(\pm\pm)}^{(ab)} + f_{(\pm\pm)}^{[ab]}. \quad (5.15)$$

Although it is not necessary to impose the symmetry property (3.22) for a general vector bundle  $\pi : E \rightarrow M$ , we will take a symmetric prescription

$$F_{AB}^{(\pm)a} = \frac{1}{2}(f_{(\pm\pm)}^{ab}\eta_{AB}^b + \eta_{AB}^b f_{(\pm\pm)}^{ba}) = f_{(\pm\pm)}^{(ab)}\eta_{AB}^b \quad (5.16)$$

to closely simulate the tangent bundle  $E = TM$  of  $M$  as much as possible where the connections are identified with  $SO(6)$  spin connections.<sup>5</sup> For a notational simplicity, we will omit the symmetrization symbol with respect to  $a \leftrightarrow b$ .

Now let us solve the Hermitian Yang-Mills equations on a Kähler manifold  $M$ . We know that solving the Hermitian Yang-Mills equation (5.10) is equivalent to the decomposition (3.43) of Yang-Mills field strengths given by (5.16) into the eigenspaces of the diagonalizable Hodge operator (3.42). And we showed that the decomposition (3.43) is actually equivalent to the branching of  $SU(4)$  under  $U(3) = SU(3) \times U(1)$  given by Eqs. (3.56) and (3.57). In other words, the  $SU(4)$  gauge group is reduced to  $U(3)$  by the background Kähler class  $\Omega$ .<sup>6</sup> Therefore, the Yang-Mills field strength in Eqs. (5.9) and (5.10) will be considered as  $U(3)$ -valued, i.e.,  $a, b$  run over 3, 6, 7, 8, 11, 12, 13, 14, 15 for the fundamental representation  $4$  and 3, 4, 5, 8, 9, 10, 13, 14, 15 for the anti-fundamental representation  $\bar{4}$ . To be precise,  $\eta_{AB}^a \in \{l_{AB}^{(+)\hat{a}}, n_{AB}^{(+)\hat{0}}\}$  for  $4$  and  $\bar{\eta}_{AB}^a \in \{l_{AB}^{(-)\hat{a}}, n_{AB}^{(-)\hat{0}}\}$  for  $\bar{4}$  where  $l_{AB}^{(\pm)\hat{a}}$  and  $n_{AB}^{(\pm)\hat{0}}$  are defined by Eqs. (3.30), (3.32), (3.35) and (3.37). As the background Kähler class  $\Omega$  determines a particular  $U(3) \subset SU(4)$  subgroup depending on its own representation, it is natural to consider that the

<sup>5</sup>Of course, the symmetric decomposition (5.16) greatly reduces the number of field strengths ( $225 \rightarrow 120$ ) and requires  $[F_{AB}^{(\pm)}, J_{\pm}^{AB}] = 0$  and  $f_{(\pm\pm)}^{ab}T^aT^b = \frac{1}{2}\{F_{AB}^{(\pm)}, J_{\pm}^{AB}\}$  where  $J_{\pm}^{AB}$  is given by Eq. (B.4). As it should be, the tangent bundle  $E = TM$  with  $F_{AB}^{(\pm)} = \frac{1}{2}R_{ABCD}J_{\pm}^{CD}$  automatically satisfies the conditions.

<sup>6</sup>It might be obvious from the expansion (5.16) which intertwines the  $SU(4)$  index  $a$  and  $SO(6)$  indices  $A, B$ . Indeed the different choice of Kähler classes can be parameterized by the homogeneous space  $SO(6)/U(3) = \mathbb{CP}^3$ . Eq. (B.17) shows that the space  $\mathbb{CP}^3 = SO(6)/U(3)$  can also be identified with the space of complex structure deformations [7]. This coincidence might presage the mirror symmetry.

Kähler classes in the representations  $\mathbf{4}$  and  $\overline{\mathbf{4}}$  are coming from different Kähler manifolds. (So to say, it is very unnatural that the same Kähler class simultaneously belongs to two different, inequivalent representations.) Thus we consider two Calabi-Yau manifolds  $M$  and  $\widetilde{M}$  whose background Kähler classes in each representation are given by

$$\Omega_{\mathbf{4}} = n_{AB}^{(+)\,0} E^A \wedge E^B, \quad \Omega_{\overline{\mathbf{4}}} = n_{AB}^{(-)\,0} \widetilde{E}^A \wedge \widetilde{E}^B. \quad (5.17)$$

After all, the Hermitian Yang-Mills equations, Eqs. (5.9) and (5.10), can be defined in the  $U(3)$  gauge theory. The so-called stability equation (5.9) for each representation is then reduced to the following equations

$$\overline{I}_{AB} F_{AB}^{(+)\,a} = f_{(++)}^{ab} \eta_{AB}^b \overline{I}_{AB} = 0 \quad \Leftrightarrow \quad f_{(++)}^{3a} + \frac{1}{\sqrt{3}} f_{(++)}^{8a} + \frac{1}{\sqrt{6}} f_{(++)}^{15a} = 0, \quad (5.18)$$

$$\overline{I}_{AB} F_{AB}^{(-)\,a} = f_{(--) }^{ab} \overline{\eta}_{AB}^b \overline{I}_{AB} = 0 \quad \Leftrightarrow \quad f_{(--) }^{3a} - \frac{1}{\sqrt{3}} f_{(--) }^{8a} - \frac{1}{\sqrt{6}} f_{(--) }^{15a} = 0. \quad (5.19)$$

By applying exactly the same argument as Eq. (3.65) and Eq. (3.72), one can conclude that Eqs. (5.18) and (5.19) are equivalent to the condition of the trivial first Chern-classes, i.e.,

$$F^{(+)\,0} = dA^{(+)\,0} = 0, \quad F^{(-)\,0} = dA^{(-)\,0} = 0, \quad (5.20)$$

where  $A^{(\pm)\,0}$  are defined by Eqs. (3.67) and (3.73). One can also see from Eq. (3.49) that the  $SU(3)$  basis  $\{l_{AB}^{(\pm)\,\hat{a}}\}$  definitely picks up the  $+$ -sign in Eq. (5.10) and its solution is given by

$$F_{AB}^{(\pm)\,\hat{a}} = f_{(\pm\pm)}^{\hat{a}\hat{b}} l_{AB}^{(\pm)\,\hat{b}}, \quad \hat{a}, \hat{b} = 1, \dots, 8. \quad (5.21)$$

Consequently we found that Hermitian Yang-Mills instantons are described by the  $SU(3)$  Yang-Mills gauge theory with the trivial first Chern class and they are connections in a *stable* holomorphic vector bundle  $E$  [2], i.e. solutions of Eqs. (5.9) and (5.10). This was precisely the result for Calabi-Yau manifolds. See, for example, Eqs. (3.64) and (3.75). Now it becomes clear what is the mirror relation for Hermitian Yang-Mills instantons. There are two kinds of Hermitian Yang-Mills instantons embedded in the fundamental representation  $\mathbf{4}$  and the anti-fundamental representation  $\overline{\mathbf{4}}$ . Each representation has its own cohomology classes, namely  $A^{(+)\,0} \in H^1(E, M)$  in the first Dolbeault cohomology group of  $M$  with values in the holomorphic vector bundle  $E$  and  $A^{(-)\,0} \in H^1(\overline{E}, \widetilde{M})$  in the first Dolbeault cohomology group of  $\widetilde{M}$  with values in the conjugate vector bundle  $\overline{E}$  [2]. Since the positive and negative chirality spinors of  $SO(6) \cong SU(4)$  transform as  $\mathbf{4}$  and  $\overline{\mathbf{4}}$ , one can identify the representation spaces  $\mathbf{4}$  and  $\overline{\mathbf{4}}$  with the spinors  $S_+$  and  $S_-$ , respectively. Also one can similarly map the cohomology classes to the fermion operators in Eqs. (4.20)-(4.22). Since the Kähler class  $\Omega$  reduces the gauge group to  $U(3)$  and we do not want to mix the creation and annihilation operators in the Hilbert space (3.3), it is required to work with the  $U(3) \subset SU(4)$  subgroup. Then one can show using the same argument in Section 4 that the cohomology classes in the holomorphic vector bundle in a complex representation  $\mathbf{3}$  and its complex conjugate  $\overline{\mathbf{3}}$  satisfy the mirror relation (4.27). The upshot is that the mirror symmetry for Calabi-Yau manifolds can be understood as the pair of Hermitian

Yang-Mills instantons in the fundamental representation  $\mathbf{3}$  and the anti-fundamental representation  $\bar{\mathbf{3}}$  of  $SU(3) \subset U(3)$ . And the Hermitian Yang-Mills instantons in different representations will get an opposite sign in the Euler characteristic  $\chi(M)$  according to the result (4.30), as will be verified below.

Since we are trying to understand the six-dimensional Euclidean gravity in terms of  $SU(4)$  gauge theory, it will be useful to calculate the Chern classes of the vector bundle  $E$  to elucidate the mirror pair between Hermitian Yang-Mills instantons. We have already shown that the first Chern class  $c_1(E)$  of the holomorphic vector bundle satisfying Eqs. (5.9) and (5.10) is trivial, i.e.  $c_1(E) = 0$ . Also we showed that the last term in Eq. (5.6) is a topological term which contains the second Chern class  $c_2(E)$ . After using Eqs. (5.9) and (5.10), one can derive the inequality

$$\frac{1}{8\pi^2} \int_M \text{Tr} F^{(+)} \wedge F^{(+)} \wedge \Omega_4 \geq 0, \quad (5.22)$$

$$\frac{1}{8\pi^2} \int_{\widetilde{M}} \text{Tr} F^{(-)} \wedge F^{(-)} \wedge \Omega_{\bar{4}} \geq 0, \quad (5.23)$$

or in a compact form,

$$\int_M c_2(E) \wedge \Omega \geq 0 \quad (5.24)$$

where  $F^{(+)} = F^{(+a)} T^a$  and  $F^{(-)} = F^{(-a)} (T^a)^*$ . This is known as the Bogomolov inequality [13, 14], which is true for all stable bundles with  $c_1(E) = 0$ . Using the identification in Eqs. (3.13) and (3.14), one may translate the above inequality into the one in gravity theory

$$-\frac{1}{16\pi^2} \int_M R_{AB} \wedge R_{AB} \wedge \Omega \geq 0. \quad (5.25)$$

Finally, the integral of the third Chern class  $c_3(E)$  in  $SU(3)$  gauge theory is given by

$$\begin{aligned} \chi(E) &= -\frac{i}{24\pi^3} \int_M \text{Tr} F^{(+)} \wedge F^{(+)} \wedge F^{(+)} \\ &= -\frac{1}{96\pi^3} \int_M d^{abc} F^{(+a)} \wedge F^{(+b)} \wedge F^{(+c)} \\ &= -\frac{1}{192\pi^3} \int_M d^6 x \sqrt{g} d^{abc} d^{def} f_{(++)}^{ad} f_{(++)}^{be} f_{(++)}^{cf} \end{aligned} \quad (5.26)$$

for the fundamental representation  $\mathbf{3}$  and

$$\begin{aligned} \chi(\bar{E}) &= -\frac{i}{24\pi^3} \int_{\widetilde{M}} \text{Tr} F^{(-)} \wedge F^{(-)} \wedge F^{(-)} \\ &= \frac{1}{96\pi^3} \int_{\widetilde{M}} d^{abc} F^{(-a)} \wedge F^{(-b)} \wedge F^{(-c)} \\ &= -\frac{1}{192\pi^3} \int_{\widetilde{M}} d^6 x \sqrt{\widetilde{g}} d^{abc} d^{def} f_{(--) }^{ad} f_{(--) }^{be} f_{(--) }^{cf} \end{aligned} \quad (5.27)$$

for the anti-fundamental representation  $\bar{\mathbf{3}}$ . It might be remarked that the relative sign  $c_3(\bar{E}) = -c_3(E)$  for the third Chern classes of a complex vector bundle  $E$  and its conjugate bundle  $\bar{E}$  arises from the

the property (5.14). One can easily check that  $\chi(E) = -\chi(\overline{E})$  if  $f_{(++)}^{ab} = -f_{(--)}^{ab}$  and  $\text{vol}(g) = \text{vol}(\widetilde{g}) = \sqrt{g}d^6x$ . This result correctly reproduces the previous one for the Calabi-Yau manifold  $M$  and its mirror manifold  $\widetilde{M}$  given by Eqs. (4.34) and (4.35). One may notice that the sign flip in the Euler characteristic is also consistent with the general result (4.39).

In conclusion, we have confirmed that the mirror pair of Calabi-Yau manifolds can be understood as the pair of Hermitian Yang-Mills instantons in the fundamental representation  $\mathbf{3}$  and the anti-fundamental representation  $\overline{\mathbf{3}}$  from the gauge theory point of view.

## 6 Discussion

Mirror symmetry of Calabi-Yau manifolds is a crucial ingredient for various string dualities. Strominger, Yau and Zaslow proposed [15] that the mirror symmetry is a T-duality transformation along dual special Lagrangian tori fibrations on mirror Calabi-Yau manifolds. It is known [15] that the T-duality transformation along the dual three-tori introduces a sign flip in the Euler characteristic as even and odd forms exchange their role. Since the odd number of T-duality operations transforms type IIB string theory to type IIA string theory and vice versa, the six-dimensional chirality of two Calabi-Yau manifolds being mirror each other will be flipped after the T-duality because the ten-dimensional chirality is correlated with the six-dimensional one. Maybe our result confirms in a different context that mirror symmetry originates from the two different chiral representations of Calabi-Yau manifolds.

Via the gauge theory formulation of six-dimensional Euclidean gravity, we could show that there are two kinds of Hermitian Yang-Mills instantons in the fundamental and anti-fundamental representations of  $SU(4) \cong SO(6)$ . Since a Calabi-Yau manifold is equivalent to a Hermitian Yang-Mills instanton from gauge theory point of view (see the quotation in the Introduction) and the chiral spinor representation of  $SO(6)$  can be identified with the (anti-)fundamental representation of  $SU(4)$ , the structure in the diagram (1.3) has been naturally anticipated. In this correspondence, the  $SU(3)$  holonomy group of a Calabi-Yau manifold is realized as the  $SU(3)$  gauge group in Yang-Mills gauge theory. Therefore, the mirror symmetry of Calabi-Yau manifolds could be understood as the existence of the *mirror* pair of Hermitian Yang-Mills instantons embedded in the fundamental and anti-fundamental representations of  $SU(3) \subset SU(4)$ .

Our gauge theory formulation of six-dimensional Euclidean gravity suggests that the existence of mirror pairs for Calabi-Yau manifolds may be generalized to general six-dimensional Riemannian manifolds like as the four-dimensional case [8]. One can consider two general Riemannian manifolds  $M$  and  $\widetilde{M}$ , for example, described by the Strominger system [18] for non-Kähler complex manifolds. The existence of two chirality classes  $\mathbb{A}$  and  $\mathbb{B}$  in Section 3 is simply a consequence of the fact that the six-dimensional Lorentz group  $SO(6)$  has two (chiral and anti-chiral) irreducible spinor representations and the  $SO(6)$  group is isomorphic to  $SU(4)/\mathbb{Z}_2$ . Therefore, one may consider a pair of metrics, for instance, given by Eq. (4.1) in different chiral representations of  $SO(6)$ . If the pairs are properly chosen such that their curvature tensors in Eqs. (3.18) and (3.19) satisfy the relation  $f_{(++)}^{ab} = -f_{(--)}^{AB}$ ,

the sign flip in the Euler characteristics (4.34) and (4.35) may be true even for general Riemannian manifolds. Furthermore, one may similarly formulate the pairing between general Riemannian manifolds in different chiral representations using a purely gauge theory approach as in Section 5. One can consider an  $SU(4)$  vector bundle  $E$  over a Riemannian manifold  $M$  and Hermitian Yang-Mills equations on  $(E, M)$  with a general Hermitian form  $\Omega$ , as constructed in [13, 19]. As might be indicated by Eqs. (5.26) and (5.27), it may be possible to find a pair of Hermitian Yang-Mills instantons satisfying  $\chi(E) = -\chi(\overline{E})$  in  $SU(4)$  Yang-Mills gauge theory. This sign flip can also be consistent with the general result (4.39). Therefore, it will be very interesting to investigate a generalization of mirror symmetry beyond Calabi-Yau manifolds. Some progress along this line will be reported elsewhere.

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## A $SO(6)$ and $SU(4)$

We will take an irreducible (Weyl) spinor representation of  $SO(6)$  whose Lorentz generators are defined by

$$J_{\pm}^{AB} \equiv \frac{1}{2}(1 \pm \Gamma^7)J^{AB} \quad (\text{A.1})$$

where  $\Gamma^7 = i\Gamma^1 \cdots \Gamma^6$ . Note that  $J_+^{AB}$  and  $J_-^{AB}$  independently satisfy the Lorentz algebra (2.3) and commute each other, i.e.,  $[J_+^{AB}, J_-^{CD}] = 0$ . They also satisfy the anti-commutation relation

$$\{J_{\pm}^{AB}, J_{\pm}^{CD}\} = -\frac{1}{2}(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}) \pm \frac{i}{2}\varepsilon^{ABCDEF}J_{\pm}^{EF}. \quad (\text{A.2})$$

Because the chiral matrix  $\Gamma^7$  is given by

$$\Gamma^7 = \begin{pmatrix} \mathbf{I}_4 & 0 \\ 0 & -\mathbf{I}_4 \end{pmatrix} \quad (\text{A.3})$$

where  $\mathbf{I}_4$  is a  $4 \times 4$  identity matrix, the Weyl spinor representation of the generators in Eq. (A.1) will be given by  $4 \times 4$  matrices. We will use the following representation of Dirac matrices [16]

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \gamma^4 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\gamma^1\gamma^2\gamma^3\gamma^4, \end{aligned} \quad (\text{A.4})$$

satisfying the  $SO(5)$  Clifford algebra relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}, \quad i, j = 1, \dots, 5. \quad (\text{A.5})$$

Then the two independent Weyl spinor representations of  $SO(6)$  are given by

$$J_+^{AB} = \{J_+^{ij} = \frac{1}{4}[\gamma^i, \gamma^j], J_+^{i6} = \frac{i}{2}\gamma^i\}, \quad (\text{A.6})$$

$$J_-^{AB} = \{J_-^{ij} = \frac{1}{4}[\gamma^i, \gamma^j], J_-^{i6} = -\frac{i}{2}\gamma^i\}. \quad (\text{A.7})$$

One can verify that the generators  $J_+^{AB}$  and  $J_-^{AB}$  separately satisfy the Lorentz algebra (2.3).

One can exchange the positive-chirality, (A.6), and negative-chirality, (A.7), representations of  $SO(6)$  by a parity transformation, a reflection  $x^M \rightarrow -x^M$  of any one element of the fundamental

six-dimensional representation of  $SO(6)$  [1]; in our case,  $x^6 \rightarrow -x^6$ . But they cannot be connected by any  $SO(6)$  rotations.

The anti-Hermitian  $4 \times 4$  matrices  $T^a = \frac{i}{2}\lambda_a$ ,  $a = 1, \dots, 15$  with vanishing traces constitute the basis of  $SU(4)$  Lie algebra. The Hermitian  $4 \times 4$  matrices  $\lambda_a$  are given by

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
\lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \lambda_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & \lambda_{15} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \quad (\text{A.8})
\end{aligned}$$

The generators  $T^a = \frac{i}{2}\lambda_a$  satisfy the following relation

$$T^a T^b = -\frac{1}{8}\delta^{ab}\mathbf{I}_4 - \frac{1}{2}f^{abc}T^c + \frac{i}{2}d^{abc}T^c \quad (\text{A.9})$$

where the structure constants  $f^{abc}$  are completely antisymmetric while  $d^{abc}$  are symmetric with respect to all of their indices. Their values are shown in the tables 1 and 2. (We get these tables from [17].)



$a$	$b$	$c$	$f^{abc}$	$a$	$b$	$c$	$f^{abc}$	$a$	$b$	$c$	$f^{abc}$
1	2	3	1	3	6	7	$-\frac{1}{2}$	6	12	13	$-\frac{1}{2}$
1	4	7	$\frac{1}{2}$	3	9	10	$\frac{1}{2}$	7	11	13	$\frac{1}{2}$
1	5	6	$-\frac{1}{2}$	3	11	12	$-\frac{1}{2}$	7	12	14	$\frac{1}{2}$
1	9	12	$\frac{1}{2}$	4	5	8	$\frac{\sqrt{3}}{2}$	8	9	10	$\frac{1}{2\sqrt{3}}$
1	10	11	$-\frac{1}{2}$	4	9	14	$\frac{1}{2}$	8	11	12	$\frac{1}{2\sqrt{3}}$
2	4	6	$\frac{1}{2}$	4	10	13	$-\frac{1}{2}$	8	13	14	$-\frac{1}{\sqrt{3}}$
2	5	7	$\frac{1}{2}$	5	9	13	$\frac{1}{2}$	9	10	15	$\sqrt{\frac{2}{3}}$
2	9	11	$\frac{1}{2}$	5	10	14	$\frac{1}{2}$	11	12	15	$\sqrt{\frac{2}{3}}$
2	10	12	$\frac{1}{2}$	6	7	8	$\frac{\sqrt{3}}{2}$	13	14	15	$\sqrt{\frac{2}{3}}$
3	4	5	$\frac{1}{2}$	6	11	14	$\frac{1}{2}$				

Table 1: The nonvanishing structure constants  $f^{abc}$

## B Six-dimensional 't Hooft symbols

The matrix representation of the six-dimensional 't Hooft symbol  $\eta_{AB}^a = -\text{Tr}(T^a J_+^{AB})$  is given by

$$\eta_{AB}^1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^1 \otimes \lambda_2,$$

$$\eta_{AB}^2 = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{i}{2} \sigma^3 \otimes \lambda_2,$$

$a$	$b$	$c$	$d^{abc}$	$a$	$b$	$c$	$d^{abc}$	$a$	$b$	$c$	$d^{abc}$
1	1	8	$\frac{1}{\sqrt{3}}$	3	9	9	$\frac{1}{2}$	7	11	14	$-\frac{1}{2}$
1	1	15	$\frac{1}{\sqrt{6}}$	3	10	10	$\frac{1}{2}$	7	12	13	$\frac{1}{2}$
1	4	6	$\frac{1}{2}$	3	11	11	$-\frac{1}{2}$	8	8	8	$-\frac{1}{\sqrt{3}}$
1	5	7	$\frac{1}{2}$	3	12	12	$-\frac{1}{2}$	8	8	15	$\frac{1}{\sqrt{6}}$
1	9	11	$\frac{1}{2}$	4	4	8	$-\frac{1}{2\sqrt{3}}$	8	9	9	$\frac{1}{2\sqrt{3}}$
1	10	12	$\frac{1}{2}$	4	4	15	$\frac{1}{\sqrt{6}}$	8	10	10	$\frac{1}{2\sqrt{3}}$
2	2	8	$\frac{1}{\sqrt{3}}$	4	9	13	$\frac{1}{2}$	8	11	11	$\frac{1}{2\sqrt{3}}$
2	2	15	$\frac{1}{\sqrt{6}}$	4	10	14	$\frac{1}{2}$	8	12	12	$\frac{1}{2\sqrt{3}}$
2	4	7	$-\frac{1}{2}$	5	5	8	$-\frac{1}{2\sqrt{3}}$	8	13	13	$-\frac{1}{\sqrt{3}}$
2	5	6	$\frac{1}{2}$	5	5	15	$\frac{1}{\sqrt{6}}$	8	14	14	$-\frac{1}{\sqrt{3}}$
2	9	12	$-\frac{1}{2}$	5	9	14	$-\frac{1}{2}$	9	9	15	$-\frac{1}{\sqrt{6}}$
2	10	11	$\frac{1}{2}$	5	10	13	$\frac{1}{2}$	10	10	15	$-\frac{1}{\sqrt{6}}$
3	3	8	$\frac{1}{\sqrt{3}}$	6	6	8	$-\frac{1}{2\sqrt{3}}$	11	11	15	$-\frac{1}{\sqrt{6}}$
3	3	15	$\frac{1}{\sqrt{6}}$	6	6	15	$\frac{1}{\sqrt{6}}$	12	12	15	$-\frac{1}{\sqrt{6}}$
3	4	4	$\frac{1}{2}$	6	11	13	$\frac{1}{2}$	13	13	15	$-\frac{1}{\sqrt{6}}$
3	5	5	$\frac{1}{2}$	6	12	14	$\frac{1}{2}$	14	14	15	$-\frac{1}{\sqrt{6}}$
3	6	6	$-\frac{1}{2}$	7	7	8	$-\frac{1}{2\sqrt{3}}$	15	15	15	$-\sqrt{\frac{2}{3}}$
3	7	7	$-\frac{1}{2}$	7	7	15	$\frac{1}{\sqrt{6}}$				

Table 2: The nonvanishing structure constants  $d^{abc}$

$$\eta_{AB}^3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^2 \otimes \left( \frac{2}{3} \mathbf{I}_3 + \frac{1}{\sqrt{3}} \lambda_8 \right),$$

$$\eta_{AB}^4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^1 \otimes \lambda_7,$$

$$\eta_{AB}^5 = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = -\frac{i}{2} \sigma^3 \otimes \lambda_7,$$

$$\eta_{AB}^6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^2 \otimes \lambda_4,$$

$$\eta_{AB}^7 = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{i}{2} \mathbf{I}_2 \otimes \lambda_5,$$

$$\eta_{AB}^8 = -\frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} = -\frac{i}{2\sqrt{3}} \sigma^2 \otimes \left( -\frac{2}{3} \mathbf{I}_3 + \lambda_3 + \frac{2}{\sqrt{3}} \lambda_8 \right),$$

$$\eta_{AB}^9 = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{i}{2} \sigma^1 \otimes \lambda_5,$$

$$\eta_{AB}^{10} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^3 \otimes \lambda_5,$$

$$\eta_{AB}^{11} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^2 \otimes \lambda_6,$$

$$\eta_{AB}^{12} = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} = -\frac{i}{2} \mathbf{I}_2 \otimes \lambda_7,$$

$$\eta_{AB}^{13} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \sigma^2 \otimes \lambda_1,$$

$$\eta_{AB}^{14} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2} \mathbf{I}_2 \otimes \lambda_2,$$

$$\eta_{AB}^{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} = \frac{i}{\sqrt{6}} \sigma^2 \otimes \left( \frac{1}{3} \mathbf{I}_3 + \lambda_3 - \frac{1}{\sqrt{3}} \lambda_8 \right),$$

where  $\mathbf{I}_2$  and  $\mathbf{I}_3$  are two and three dimensional identity matrices, respectively, and  $(\sigma^1, \sigma^2, \sigma^3)$  are the Pauli matrices and  $\lambda_a$  ( $a = 1, \dots, 8$ ) are the  $SU(3)$  Gell-Mann matrices.

The matrix representation of the six-dimensional 't Hooft symbol  $\bar{\eta}_{AB}^a = -\text{Tr}(T^a J_-^{AB})$  can be obtained by flipping a sign in the sixth row and column of the matrix  $\eta_{AB}^a$ :

$$\begin{aligned} \bar{\eta}_{AB}^1 &= \frac{i}{2} \sigma^1 \otimes \lambda_2, & \bar{\eta}_{AB}^2 &= -\frac{i}{2} \sigma^3 \otimes \lambda_2, & \bar{\eta}_{AB}^3 &= \frac{i}{2} \sigma^2 \otimes \left( \frac{2}{3} \mathbf{I}_3 + \frac{1}{\sqrt{3}} \lambda_8 \right), \\ \bar{\eta}_{AB}^4 &= -\frac{i}{2} \sigma^2 \otimes \lambda_6, & \bar{\eta}_{AB}^5 &= -\frac{i}{2} \mathbf{I}_2 \otimes \lambda_7, & \bar{\eta}_{AB}^6 &= -\frac{i}{2} \sigma^1 \otimes \lambda_5, \\ \bar{\eta}_{AB}^7 &= -\frac{i}{2} \sigma^3 \otimes \lambda_5, & \bar{\eta}_{AB}^8 &= -\frac{i}{2\sqrt{3}} \sigma^2 \otimes \left( \frac{2}{3} \mathbf{I}_3 + \lambda_3 - \frac{2}{\sqrt{3}} \lambda_8 \right), & \bar{\eta}_{AB}^9 &= \frac{i}{2} \sigma^2 \otimes \lambda_4, \\ \bar{\eta}_{AB}^{10} &= \frac{i}{2} \mathbf{I}_2 \otimes \lambda_5, & \bar{\eta}_{AB}^{11} &= -\frac{i}{2} \sigma^1 \otimes \lambda_7, & \bar{\eta}_{AB}^{12} &= -\frac{i}{2} \sigma^3 \otimes \lambda_7, \\ \bar{\eta}_{AB}^{13} &= \frac{i}{2} \sigma^2 \otimes \lambda_1, & \bar{\eta}_{AB}^{14} &= \frac{i}{2} \mathbf{I}_2 \otimes \lambda_2, & \bar{\eta}_{AB}^{15} &= \frac{i}{\sqrt{6}} \sigma^2 \otimes \left( -\frac{1}{3} \mathbf{I}_3 + \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right). \end{aligned} \tag{B.1}$$

In order to derive algebras obeyed by the 't Hooft symbols, first note that either the  $SO(6)$  generators  $J_{\pm}^{AB}$  or the  $SU(4)$  generators  $T^a$  can serve as a complete basis of any traceless, Hermitian  $4 \times 4$  matrix  $K$ , i.e.,

$$K = \sum_{a=1}^{15} k_a T^a = \frac{1}{2} \sum_{A,B=1}^6 K_{AB} J_{\pm}^{AB}. \tag{B.2}$$

Using the relation (3.15), one can easily deduce that

$$T^a = \frac{1}{2} \eta_{AB}^a J_+^{AB} = \frac{1}{2} \bar{\eta}_{AB}^a J_-^{AB}, \tag{B.3}$$

$$J_+^{AB} = 2 \eta_{AB}^a T^a, \quad J_-^{AB} = 2 \bar{\eta}_{AB}^a T^a, \tag{B.4}$$

assuming the summation convention. Then one can consider the following matrix products

$$\text{I : } T^a T^b = \frac{1}{4} \eta_{AB}^a \eta_{CD}^b J_+^{AB} J_+^{CD}, \quad J_+^{AB} J_+^{CD} = 4 \eta_{AB}^a \eta_{CD}^b T^a T^b, \quad (\text{B.5})$$

$$\text{II : } T^a T^b = \frac{1}{4} \bar{\eta}_{AB}^a \bar{\eta}_{CD}^b J_-^{AB} J_-^{CD}, \quad J_-^{AB} J_-^{CD} = 4 \bar{\eta}_{AB}^a \bar{\eta}_{CD}^b T^a T^b. \quad (\text{B.6})$$

Using Eqs. (2.3), (A.2) and (A.9), one can easily get the algebras obeyed by the six-dimensional 't Hooft symbols:

$$\eta_{AB}^a \eta_{AB}^b = \delta^{ab} = \bar{\eta}_{AB}^a \bar{\eta}_{AB}^b, \quad (\text{B.7})$$

$$\eta_{AB}^a \eta_{CD}^a = \frac{1}{2} (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}) = \bar{\eta}_{AB}^a \bar{\eta}_{CD}^a, \quad (\text{B.8})$$

$$\frac{1}{4} \varepsilon^{ABCDEF} \eta_{CD}^a \eta_{EF}^b = d^{abc} \eta_{AB}^c, \quad (\text{B.9})$$

$$\frac{1}{4} \varepsilon^{ABCDEF} \bar{\eta}_{CD}^a \bar{\eta}_{EF}^b = -d^{abc} \bar{\eta}_{AB}^c, \quad (\text{B.10})$$

$$\eta_{AC}^a \eta_{BC}^b - \eta_{BC}^a \eta_{AC}^b = f^{abc} \eta_{AB}^c, \quad (\text{B.11})$$

$$\bar{\eta}_{AC}^a \bar{\eta}_{BC}^b - \bar{\eta}_{BC}^a \bar{\eta}_{AC}^b = f^{abc} \bar{\eta}_{AB}^c, \quad (\text{B.12})$$

$$f^{abc} \eta_{AB}^a \eta_{CD}^b = \frac{1}{2} (\delta_{AC} \eta_{BD}^c - \delta_{AD} \eta_{BC}^c - \delta_{BC} \eta_{AD}^c + \delta_{BD} \eta_{AC}^c), \quad (\text{B.13})$$

$$f^{abc} \bar{\eta}_{AB}^a \bar{\eta}_{CD}^b = \frac{1}{2} (\delta_{AC} \bar{\eta}_{BD}^c - \delta_{AD} \bar{\eta}_{BC}^c - \delta_{BC} \bar{\eta}_{AD}^c + \delta_{BD} \bar{\eta}_{AC}^c), \quad (\text{B.14})$$

$$d^{abc} \eta_{AB}^a \eta_{CD}^b = \frac{1}{4} \varepsilon^{ABCDEF} \eta_{EF}^c, \quad (\text{B.15})$$

$$d^{abc} \bar{\eta}_{AB}^a \bar{\eta}_{CD}^b = -\frac{1}{4} \varepsilon^{ABCDEF} \bar{\eta}_{EF}^c. \quad (\text{B.16})$$

Finally we list the nonzero components of the 't Hooft symbols in the basis of complex coordinates  $z^\alpha = \{z^1 = x^1 + ix^2, z^2 = x^3 + ix^4, z^3 = x^5 + ix^6\}$  and their complex conjugates  $\bar{z}^{\bar{\alpha}}$  where  $\alpha, \bar{\alpha} = 1, 2, 3$ . We will denote  $\eta_{\alpha\beta}^a = \eta_{z^\alpha z^\beta}^a$ ,  $\eta_{\alpha\bar{\beta}}^a = \eta_{z^\alpha \bar{z}^\beta}^a$ , etc. in the hope of no confusion with the real basis:

$$\begin{aligned} \eta_{12}^1 &= -\frac{i}{4}, & \eta_{12}^2 &= -\frac{1}{4}, & \eta_{23}^4 &= -\frac{i}{4}, & \eta_{23}^5 &= -\frac{1}{4}, & \eta_{13}^9 &= \frac{i}{4}, & \eta_{13}^{10} &= \frac{1}{4}, \\ \eta_{11}^3 &= \frac{i}{4}, & \eta_{22}^3 &= \frac{i}{4}, & \eta_{13}^6 &= \frac{i}{4}, & \eta_{13}^7 &= -\frac{1}{4}, & & & & \\ \eta_{11}^8 &= -\frac{i}{4\sqrt{3}}, & \eta_{22}^8 &= \frac{i}{4\sqrt{3}}, & \eta_{33}^8 &= \frac{i}{2\sqrt{3}}, & \eta_{23}^{11} &= \frac{i}{4}, & \eta_{23}^{12} &= -\frac{1}{4}, & & \\ \eta_{12}^{13} &= \frac{i}{4}, & \eta_{12}^{14} &= \frac{1}{4}, & \eta_{11}^{15} &= \frac{i}{2\sqrt{6}}, & \eta_{22}^{15} &= -\frac{i}{2\sqrt{6}}, & \eta_{33}^{15} &= \frac{i}{2\sqrt{6}}. & & \end{aligned} \quad (\text{B.17})$$

Here the complex conjugates are not shown up since they can easily be implemented. The corresponding values of  $\bar{\eta}_{AB}^a$  can be obtained from those of  $\eta_{AB}^a$  by interchanging  $z^3 \leftrightarrow \bar{z}^3$ . The above result implies that the space of complex structure deformations can be identified with the homogeneous space  $SO(6)/U(3) = \mathbb{C}P^3$  [7].

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